

Littlewood-Paley Characterizations of Anisotropic Hardy-Lorentz Spaces

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Abstract Let $p \in (0, 1]$, $q \in (0, \infty]$ and A be a general expansive matrix on \mathbb{R}^n . Let $H_A^{p,q}(\mathbb{R}^n)$ be the anisotropic Hardy-Lorentz spaces associated with A defined via the non-tangential grand maximal function. In this article, the authors characterize $H_A^{p,q}(\mathbb{R}^n)$ in terms of the Lusin-area function, the Littlewood-Paley g -function or the Littlewood-Paley g_λ^* -function via first establishing an anisotropic Fefferman-Stein vector-valued inequality in the Lorentz space $L^{p,q}(\mathbb{R}^n)$. All these characterizations are new even for the classical isotropic Hardy-Lorentz spaces on \mathbb{R}^n . Moreover, the range of λ in the g_λ^* -function characterization of $H_A^{p,q}(\mathbb{R}^n)$ coincides with the best known one in the classical Hardy space $H^p(\mathbb{R}^n)$ or in the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$.

1 Introduction

It is well known that $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, particularly, in the study for the boundedness of maximal functions and singular integral operators. Moreover, when studying the boundedness of these operators in the critical case, the weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ naturally appears and have been proved to be a good substitute of $H^p(\mathbb{R}^n)$. For example, it is known that, if T is a δ -type Calderón-Zygmund operator with $\delta \in (0, 1]$ and $T^*(1) = 0$, where T^* denotes the *adjoint operator* of T , then T is bounded on $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+\delta}, 1]$ (see, for example, [6]), but T is not bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$; while Liu [50] proved that T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$. Liu [50] also obtained the ∞ -atomic decomposition of $WH^p(\mathbb{R}^n)$ for all $p \in (0, 1]$. Recall that the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ with $p \in (0, 1)$ were first introduced by Fefferman, Rivi re and Sagher [31] in 1974, which naturally appears as the intermediate spaces of Hardy spaces $H_A^p(\mathbb{R}^n)$ with $p \in (0, 1]$ via the real interpolation. Later on, the weak Hardy space $H^{1,\infty}(\mathbb{R}^n)$ was introduced by Fefferman and Soria [35] to find out the biggest space from which the Hilbert transform is bounded to the weak Lebesgue space $L^{1,\infty}(\mathbb{R}^n)$, meanwhile, they also obtained the ∞ -atomic decomposition of $H^{1,\infty}(\mathbb{R}^n)$ and the boundedness of some Calder n-Zygmund operators from $H^{1,\infty}(\mathbb{R}^n)$ to

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$L^{1,\infty}(\mathbb{R}^n)$. In 1994, Álvarez [4] studied the Calderón-Zygmund theory related to $H^{p,\infty}(\mathbb{R}^n)$ with $p \in (0, 1]$. Nowadays, it is well known that the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$, with $p \in (0, 1]$, play a key role when studying the boundedness of operators in the critical case; see, for example, [4, 5, 28, 38, 29, 72, 30]. Moreover, the weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n)$ are also known as special cases of the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ which, when $p = 1$ and $q \in (1, \infty)$, were introduced and studied by Parilov [58]. For the full range $p \in (0, 1]$ and $q \in (0, \infty]$, the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ were investigated by Abu-Shammala and Torchinsky [1], and their ∞ -atomic characterizations, real interpolation properties over parameter q , and the boundedness of singular integrals and some other operators on these spaces were also presented. In 2010, Almeida and Caetano [3] studied the generalized Hardy spaces, which include the classical Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^n)$ (see [1]) as special cases, in which, they obtained some maximal characterizations and real interpolation results of these generalized Hardy spaces and, as applications, they proved the boundedness of some classical operators in this generalized setting.

The Lorentz spaces were originally studied by Lorentz [52, 53] in the early 1950's. As a generalization of $L^p(\mathbb{R}^n)$, Lorentz spaces are known as intermediate spaces of Lebesgue spaces in the real interpolation method; see [16, 48, 59]. For a systematic treatment of Lorentz spaces and their dual spaces, we refer the reader to Hunt [41], Cwikel [24] and Cwikel and Fefferman [25, 26]; see also [8, 9, 39, 62, 67]. It is well known that, due to their fine structures, Lorentz spaces play an irreplaceable role in the study on various critical or endpoint analysis problems from many different research fields and there exists a lot of literatures on this subject, here we only mention several recent papers from harmonic analysis (see, for example, [57, 56, 64, 68]) and partial differential equations (see, for example, [42, 55, 60]).

With the enlightening work of Stein and Weiss [66] on systems of conjugate harmonic functions, higher-dimensional extensions of Hardy spaces naturally appear. At the same time, a series of characterizations of Hardy spaces were obtained one after another; see [14, 54, 65]. Recall that the original work of the Littlewood-Paley theory should be owed to Littlewood and Paley [49]. Moreover, the Littlewood-Paley theory of Hardy spaces were investigated by Calderón [17] and Fefferman and Stein [33]. In addition, the Littlewood-Paley theory of other useful function spaces, for example, various forms of the Lipschitz spaces, the space $BMO(\mathbb{R}^n)$ and Sobolev spaces, has also been well developed, which provides one of the most successful unifying perspectives on these spaces (see [37]).

On the other hand, from 1970's, there has been an increasing interest in extending classical function spaces arising in harmonic analysis from Euclidean spaces to anisotropic settings and some other domains; see, for example, [19, 20, 36, 63, 69, 70]. The study of function spaces on \mathbb{R}^n associated with anisotropic dilations dates from these celebrated articles [18, 19, 20] of Calderón and Torchinsky on anisotropic Hardy spaces. Since then, the theory of anisotropic function spaces was well developed by many authors; see, for example, [36, 65, 69]. In 2003, Bownik [10] introduced and investigated the anisotropic Hardy spaces associated with general expansive matrixes, which were extended to the weighted setting in [13]. For further efforts of function spaces and related operators on the anisotropic Euclidean spaces, we refer the reader to [11, 12, 13, 27, 43, 44, 45, 71].

Moreover, the authors [51] introduced the anisotropic Hardy-Lorentz spaces $H_A^{p,q}(\mathbb{R}^n)$

and obtained some characterizations of these spaces; for example, characterizations in terms of the atoms or the molecules, the radial or the non-tangential maximal functions or the finite atomic decomposition and, also considered some interpolation properties of the anisotropic Hardy-Lorentz spaces $H_A^{p,q}(\mathbb{R}^n)$ via the real method and the boundedness of some classical operators in this anisotropic setting. To further complete the theory of the anisotropic Hardy-Lorentz spaces, in this article, we establish the characterizations of $H_A^{p,q}(\mathbb{R}^n)$ via Littlewood-Paley functions including the Lusin-area function, the Littlewood-Paley g -function or the Littlewood-Paley g_λ^* -function.

To be precise, this article is organized as follows.

In Section 2, we first present some basic notions and notation used in this article, including Lorentz spaces and their properties and also some known facts on expansive matrixes in [10]. Then we recall the definitions of the anisotropic Hardy-Lorentz spaces via non-tangential grand maximal functions (denoted by $H_A^{p,q}(\mathbb{R}^n)$) and their atomic variants (denoted by $H_A^{p,r,s,q}(\mathbb{R}^n)$). Moreover, we state the main results of this article, namely, the characterizations of $H_A^{p,q}(\mathbb{R}^n)$ in terms of the Lusin-area function (see Theorem 2.11 below), the Littlewood-Paley g -function (see Theorem 2.12 below) or the Littlewood-Paley g_λ^* -function (see Theorem 2.13 below). We point out that all these characterizations are new even for the classical isotropic Hardy-Lorentz spaces on \mathbb{R}^n .

In Section 3, by the anisotropic Calderón reproducing formula and the method used in the proof of the atomic or the molecular characterizations of $H_A^{p,q}(\mathbb{R}^n)$ (see [51]), we give out the proof of Theorem 2.11. We point out that, when we decompose a distribution into a sum of atoms, the dual method used in estimating each atom in the classic case does not work any more in the present setting. Instead, we use a method from Fefferman [34] to obtain a subtle estimate (see (3.23) below).

In Section 4, via the above Lusin area function characterizations (namely, Theorem 2.11), we first show Theorem 2.12. To this end, via borrowing some ideas from [73, Theorem 1], we establish an anisotropic Fefferman-Stein vector-valued inequality in Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ (see Lemma 4.5 below) which plays a key role in the proof of Theorem 2.12 and is of independent interest. Besides this, we also employ the discrete Calderón reproducing formula from [46, Lemma 3.2], which is an anisotropic version of [37, Theorem 6.16], and some auxiliary inequalities (see Lemmas 4.7 and 4.8 below). The proof of Lemma 4.8 borrows some ideas from [12, Lemma 3.3]. We also point out that the method used in the proof of Theorem 2.12 is different from that used by Liang et al. in the proof of [47, Theorem 4.4], in which a subtle pointwise upper estimate via the vector-valued Hardy-Littlewood maximal function (see [47, (4.2)]) plays a key role. Moreover, motivated by the proof of [46, Theorem 3.9], together with using some ideas from Folland and Stein [36, Theorem 7.1] and Aguilera and Segovia [2, Theorem 1], we further prove Theorem 2.13 for all $\lambda \in (2/p, \infty)$ in this section. To this end, we first prove that the $L^{p,q}(\mathbb{R}^n)$ quasi-norm of the variant of the anisotropic Lusin area function $S_{k_0}(f)$ can be controlled by the $L^{p,q}(\mathbb{R}^n)$ quasi-norm of the Lusin area function $S(f)$ for all $k_0 \in \mathbb{N}$ and $f \in L^{p,q}(\mathbb{R}^n)$ (see Lemma 4.11 below), which is a key technique used in the proof of Theorem 2.13. We point out that the range of λ in Theorem 2.13 coincides with the best known one in the classical Hardy space $H^p(\mathbb{R}^n)$ or in the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$.

Finally, we make some conventions on notation. Throughout this article, we always

let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any multi-index $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, let $|\beta| := \beta_1 + \dots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \dots (\frac{\partial}{\partial x_n})^{\beta_n}$. The symbol C denotes a *positive constant* which is independent of the main parameters, but its value may change from line to line. *Constants with subscripts*, such as C_1 , are the same in different occurrences. We also use $C_{(\alpha, \beta, \dots)}$ to denote a positive constant depending on the indicated parameters α, β, \dots . Moreover, we use $f \lesssim g$ to denote $f \leq Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For every index $r \in [1, \infty]$, we use r' to denote its *conjugate index*, namely, $1/r + 1/r' = 1$. In addition, for any set $F \subset \mathbb{R}^n$, we denote by F^c the set $\mathbb{R}^n \setminus F$, by χ_F its *characteristic function*, and by $\#F$ the cardinality of F . The symbol $\lfloor s \rfloor$, for any $s \in \mathbb{R}$, denotes the maximal integer not larger than s .

2 Main results

In this section, we recall the notion of the anisotropic Hardy-Lorentz space defined in [51], and then state our main results.

We begin with the definition of Lorentz spaces. Let $p \in (0, \infty)$ and $q \in (0, \infty]$. The *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f satisfying that $\|f\|_{L^{p,q}(\mathbb{R}^n)} < \infty$, where the quasi-norm

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \left\{ \frac{q}{p} \int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right\}^{\frac{1}{q}} & \text{when } q \in (0, \infty), \\ \sup_{t \in (0, \infty)} \left[t^{\frac{1}{p}} f^*(t) \right] & \text{when } q = \infty, \end{cases}$$

and f^* denotes the *non-increasing rearrangement* of f , namely,

$$f^*(t) := \{\alpha \in (0, \infty) : d_f(\alpha) \leq t\}, \quad t \in (0, \infty).$$

Here and hereafter, for any $\alpha \in (0, \infty)$, $d_f(\alpha) := |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$. It is well known that, if $q \in (0, \infty)$,

$$(2.1) \quad \|f\|_{L^{p,q}(\mathbb{R}^n)} \sim \left\{ \int_0^\infty \alpha^{q-1} [d_f(\alpha)]^{\frac{q}{p}} d\alpha \right\}^{\frac{1}{q}} \sim \left\{ \sum_{k \in \mathbb{Z}} \left[2^k \left\{ d_f(2^k) \right\}^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}}$$

and

$$(2.2) \quad \|f\|_{L^{p,\infty}(\mathbb{R}^n)} \sim \sup_{\alpha \in (0, \infty)} \left\{ \alpha [d_f(\alpha)]^{\frac{1}{p}} \right\} \sim \sup_{k \in \mathbb{Z}} \left\{ 2^k \left[d_f(2^k) \right]^{\frac{1}{p}} \right\};$$

see [39]. By [39, Remark 1.4.7], for any $p, r \in (0, \infty)$, $q \in (0, \infty]$ and all measurable functions g , we know that

$$(2.3) \quad \| |g|^r \|_{L^{p,q}(\mathbb{R}^n)} = \|g\|_{L^{pr,qr}(\mathbb{R}^n)}^r.$$

Now let us recall the notion of expansive matrices in [10].

Definition 2.1. An $n \times n$ real matrix A is called an *expansive matrix* (for short, a *dilation*) if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, here and hereafter, $\sigma(A)$ denotes the set of all eigenvalues of A .

Throughout this article, A always denotes a fixed dilation and $b := |\det A|$. By [10, p. 6, (2.7)], we have $b \in (1, \infty)$. Let λ_- and λ_+ be two *positive numbers* satisfying that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when A is diagonalizable over \mathbb{C} , we can even take $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments. Then, by [10, p. 5 (2.1) and (2.2)], there exists a positive constant C , independent of x and j , such that, for all $x \in \mathbb{R}^n$, when $j \in \mathbb{Z}_+$,

$$C^{-1}(\lambda_-)^j |x| \leq |A^j x| \leq C(\lambda_+)^j |x|$$

and, when $j \in \mathbb{Z} \setminus \mathbb{Z}_+$,

$$(2.4) \quad C^{-1}(\lambda_+)^j |x| \leq |A^j x| \leq C(\lambda_-)^j |x|.$$

It was proved in [10, p. 5, Lemma 2.2] that, for a given dilation A , there exists an open ellipsoid Δ and $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n -dimensional Lebesgue measure of the set Δ . Let $B_k := A^k \Delta$ for all $k \in \mathbb{Z}$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Let \mathfrak{B} be the set of all such dilated balls, namely,

$$(2.5) \quad \mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}.$$

Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. Throughout this article, let τ be the *minimal integer* such that $r^\tau \geq 2$. Then, for all $k \in \mathbb{Z}$, it holds true that

$$(2.6) \quad B_k + B_k \subset B_{k+\tau},$$

$$(2.7) \quad B_k + (B_{k+\tau})^{\mathbb{C}} \subset (B_k)^{\mathbb{C}},$$

where $E + F$ denotes the algebraic sum $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

The notion of the homogeneous quasi-norm induced by A was introduced in [10, p. 6, Definition 2.3] as follows.

Definition 2.2. A *homogeneous quasi-norm* associated with a dilation A is a measurable mapping $\rho : \mathbb{R}^n \rightarrow [0, \infty]$ satisfying that

- (i) $\rho(x) = 0 \iff x = \vec{0}_n$, here and hereafter, $\vec{0}_n := (0, \dots, 0) \in \mathbb{R}^n$;
- (ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$;
- (iii) $\rho(x + y) \leq H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of x and y .

In the standard dyadic case $A := 2\mathbf{I}_{n \times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of the homogeneous quasi-norm associated with A , here and hereafter, $\mathbf{I}_{n \times n}$ denotes the $n \times n$ unit matrix and $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^n . It was proved in [10, p. 6, Lemma 2.4] that all homogeneous quasi-norms associated with A are equivalent. Therefore, for a given dilation A , in what follows, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \begin{cases} b^j & \text{when } x \in B_{j+1} \setminus B_j, \\ 0 & \text{when } x = 0 \end{cases}$$

for convenience. Obviously, for all $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : \rho(x) < b^k\}$. Moreover, (\mathbb{R}^n, ρ, dx) is a space of homogeneous type in the sense of Coifman and Weiss [22, 23], here and hereafter, dx denotes the n -dimensional Lebesgue measure.

Denote the space of all Schwartz functions by $\mathcal{S}(\mathbb{R}^n)$, namely, the set of all $C^\infty(\mathbb{R}^n)$ functions φ satisfying that, for every integer $\ell \in \mathbb{Z}_+$ and multi-index α ,

$$\|\varphi\|_{\alpha, \ell} := \sup_{x \in \mathbb{R}^n} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty.$$

The dual space of $\mathcal{S}(\mathbb{R}^n)$, namely, the space of all tempered distributions on \mathbb{R}^n equipped with the weak-* topology, is denoted by $\mathcal{S}'(\mathbb{R}^n)$. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha, \ell} \leq 1, |\alpha| \leq N, \ell \leq N\};$$

equivalently,

$$\varphi \in \mathcal{S}_N(\mathbb{R}^n) \iff \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \left[|\partial^\alpha \varphi(x)| \max \left\{ 1, [\rho(x)]^N \right\} \right] \leq 1.$$

Throughout this article, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\varphi_k(\cdot) := b^{-k} \varphi(A^{-k} \cdot)$.

Definition 2.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_\varphi(f)$ of f with respect to φ is defined as

$$M_\varphi(f)(x) := \sup_{y \in x + B_k, k \in \mathbb{Z}} |f * \varphi_k(y)|, \quad \forall x \in \mathbb{R}^n.$$

Moreover, for $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of f is defined as

$$(2.8) \quad M_N(f)(x) := \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x), \quad \forall x \in \mathbb{R}^n.$$

The following Proposition 2.4 is just [10, p. 13, Theorem 3.6].

Proposition 2.4. For any $s \in (1, \infty)$, let

$$\mathcal{F} := \left\{ \varphi \in L^\infty(\mathbb{R}^n) : |\varphi(x)| \leq [1 + \rho(x)]^{-s}, x \in \mathbb{R}^n \right\}.$$

For $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^n)$, the maximal function associated with \mathcal{F} , $M_{\mathcal{F}}$, is defined as

$$M_{\mathcal{F}}(f)(x) := \sup_{\varphi \in \mathcal{F}} M_{\varphi}(f)(x), \quad \forall x \in \mathbb{R}^n.$$

Then there exists a positive constant $C_{(s)}$, depending on s , such that, for all $\lambda \in (0, \infty)$ and $f \in L^1(\mathbb{R}^n)$,

$$(2.9) \quad |\{x \in \mathbb{R}^n : M_{\mathcal{F}}(f)(x) > \lambda\}| \leq C_{(s)} \|f\|_{L^1(\mathbb{R}^n)} / \lambda$$

and, for all $p \in (1, \infty]$, there exists a positive constant $C_{(s,p)}$, depending on s and p , such that, for all $f \in L^p(\mathbb{R}^n)$,

$$(2.10) \quad \|M_{\mathcal{F}}(f)\|_{L^p(\mathbb{R}^n)} \leq C_{(s,p)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Remark 2.5. Obviously, by Proposition 2.4, we know that the non-tangential grand maximal function $M_N(f)$ in (2.8) and the Hardy-Littlewood maximal function $M_{\text{HL}}(f)$, defined by setting, for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$(2.11) \quad M_{\text{HL}}(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{y \in x+B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)| dz = \sup_{x \in B \in \mathfrak{B}} \frac{1}{|B|} \int_B |f(z)| dz,$$

where \mathfrak{B} is as in (2.5), satisfy (2.9) and (2.10).

The following anisotropic Hardy-Lorentz space and its atomic variant were introduced in [51].

Definition 2.6. Let $p \in (0, \infty)$, $q \in (0, \infty]$ and

$$N_{(p)} := \begin{cases} \left\lfloor \left(\frac{1}{p} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor + 2 & \text{when } p \in (0, 1], \\ 2 & \text{when } p \in (1, \infty). \end{cases}$$

For any $N \in \mathbb{N} \cap (N_{(p)}, \infty)$, the *anisotropic Hardy-Lorentz space* $H_A^{p,q}(\mathbb{R}^n)$ is defined by

$$H_A^{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : M_N(f) \in L^{p,q}(\mathbb{R}^n)\}$$

and, for any $f \in H_A^{p,q}(\mathbb{R}^n)$, let $\|f\|_{H_A^{p,q}(\mathbb{R}^n)} := \|M_N(f)\|_{L^{p,q}(\mathbb{R}^n)}$.

Definition 2.7. (i) An anisotropic triplet (p, r, s) is said to be *admissible* if $p \in (0, 1]$, $r \in (1, \infty]$ and $s \in \mathbb{N}$ with $s \geq \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor$. For an admissible anisotropic triplet (p, r, s) , a measurable function a on \mathbb{R}^n is called an *anisotropic (p, r, s) -atom* if

- (a) $\text{supp } a \subset B$, where $B \in \mathfrak{B}$ and \mathfrak{B} is as in (2.5);
- (b) $\|a\|_{L^r(\mathbb{R}^n)} \leq |B|^{1/r-1/p}$;
- (c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

(ii) For an anisotropic triplet (p, r, s) , $q \in (0, \infty]$ and a dilation A , the *anisotropic atomic Hardy-Lorentz space* $H_A^{p,r,s,q}(\mathbb{R}^n)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{a_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}}$ of (p, r, s) -atoms, respectively, supported on $\{x_i^k + B_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}} \subset \mathfrak{B}$, and a positive constant \tilde{C} such that $\sum_{i \in \mathbb{N}} \chi_{x_i^k + B_i^k}(x) \leq \tilde{C}$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and $f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\lambda_i^k \sim 2^k |B_i^k|^{1/p}$ for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$ with the implicit equivalent positive constants independent of k and i .

Moreover, for all $f \in H_A^{p,r,s,q}(\mathbb{R}^n)$, define

$$\|f\|_{H_A^{p,r,s,q}(\mathbb{R}^n)} := \inf \left\{ \left[\sum_{k \in \mathbb{Z}} \left(\sum_{i \in \mathbb{N}} |\lambda_i^k|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} : f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k \right\}$$

with the usual modification made when $q = \infty$, where the infimum is taken over all decompositions of f as above.

The following decomposition characterizations of $H_A^{p,q}(\mathbb{R}^n)$ is just [51, Theorem 3.6].

Lemma 2.8. *Let (p, r, s) be an admissible anisotropic triplet as in Definition 2.7(i), $q \in (0, \infty]$ and $N \in \mathbb{N} \cap (N_{(p)}, \infty)$. Then $H_A^{p,q}(\mathbb{R}^n) = H_A^{p,r,s,q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Remark 2.9. From Lemma 2.8, it follows that the space $H_A^{p,q}(\mathbb{R}^n)$ is independent of the choice of N as long as $N \in \mathbb{N} \cap (N_{(p)}, \infty)$.

Definition 2.10. Suppose that $p \in (0, 1]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \varphi(x) x^\alpha dx = 0$ for all multi-indices $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, where $s \in \mathbb{N}$ and $s \geq \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\lambda \in (0, \infty)$, the *anisotropic Lusin area function* $S(f)$, the *Littlewood-Paley g -function* $g(f)$ and the *Littlewood-Paley g_λ^* -function* $g_\lambda^*(f)$ are defined, respectively, by setting, for all $x \in \mathbb{R}^n$,

$$(2.12) \quad S(f)(x) := \left[\sum_{k \in \mathbb{Z}} b^{-k} \int_{x+B_k} |f * \varphi_k(y)|^2 dy \right]^{1/2},$$

$$g(f)(x) := \left[\sum_{k \in \mathbb{Z}} |f * \varphi_k(x)|^2 \right]^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left\{ \sum_{k \in \mathbb{Z}} b^{-k} \int_{\mathbb{R}^n} \left[\frac{b^k}{b^k + \rho(x-y)} \right]^\lambda |f * \varphi_k(y)|^2 dy \right\}^{1/2}.$$

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, $f * \psi_k \rightarrow 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \rightarrow \infty$. Denote by $\mathcal{S}'_0(\mathbb{R}^n)$ the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity.

The following Theorems 2.11 through 2.13 are the main results of this article, which characterize the space $H_A^{p,q}(\mathbb{R}^n)$, respectively, in terms of the Lusin area function, the Littlewood-Paley g -function and the Littlewood-Paley g_λ^* -function.

Theorem 2.11. *Let $p \in (0, 1]$ and $q \in (0, \infty]$. Then $f \in H_A^{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{p,q}(\mathbb{R}^n)$. Moreover, there exists a positive constant C_1 such that, for all $f \in H_A^{p,q}(\mathbb{R}^n)$,*

$$\frac{1}{C_1} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \leq \|f\|_{H_A^{p,q}(\mathbb{R}^n)} \leq C_1 \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

Theorem 2.12. *Let $p \in (0, 1]$ and $q \in (0, \infty]$. Then $f \in H_A^{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g(f) \in L^{p,q}(\mathbb{R}^n)$. Moreover, there exists a positive constant C_2 such that, for all $f \in H_A^{p,q}(\mathbb{R}^n)$,*

$$\frac{1}{C_2} \|g(f)\|_{L^{p,q}(\mathbb{R}^n)} \leq \|f\|_{H_A^{p,q}(\mathbb{R}^n)} \leq C_2 \|g(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

Theorem 2.13. *Let $p \in (0, 1]$, $q \in (0, \infty]$ and $\lambda \in (2/p, \infty)$. Then $f \in H_A^{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g_\lambda^*(f) \in L^{p,q}(\mathbb{R}^n)$. Moreover, there exists a positive constant C_3 such that, for all $f \in H_A^{p,q}(\mathbb{R}^n)$,*

$$\frac{1}{C_3} \|g_\lambda^*(f)\|_{L^{p,q}(\mathbb{R}^n)} \leq \|f\|_{H_A^{p,q}(\mathbb{R}^n)} \leq C_3 \|g_\lambda^*(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

Remark 2.14. We point out that the range of λ in Theorem 2.13 coincides with the best known one in the classical Hardy space $H^p(\mathbb{R}^n)$ or in the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$.

3 Proof of Theorem 2.11

From [51, Proposition 2.7], we deduce that, for all $p \in (0, \infty)$ and $q \in (0, \infty]$, $H_A^{p,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Moreover, we have the following useful property of $H_A^{p,q}(\mathbb{R}^n)$.

Proposition 3.1. *Let $p \in (0, 1]$ and $q \in (0, \infty]$. Then*

$$H_A^{p,q}(\mathbb{R}^n) \subset \mathcal{S}'_0(\mathbb{R}^n).$$

Proof. Observe that, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $f \in H_A^{p,q}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $y \in x + B_k$, $|f * \psi_k(x)| \lesssim M_N(f)(y)$. Hence, there exists a positive constant C_4 such that

$$x + B_k \subset \{y \in \mathbb{R}^n : M_N(f)(y) > C_4 |f * \psi_k(x)|\}.$$

By this and (2.2), we further have

$$\begin{aligned} |f * \psi_k(x)| &= |B_k|^{-1/p} |B_k|^{1/p} |f * \psi_k(x)| \\ &\lesssim |B_k|^{-1/p} |\{y \in \mathbb{R}^n : M_N(f)(y) > C_4 |f * \psi_k(x)|\}|^{1/p} |f * \psi_k(x)| \\ &\lesssim |B_k|^{-1/p} \|f\|_{H^{p,\infty}(\mathbb{R}^n)} \lesssim |B_k|^{-1/p} \|f\|_{H_A^{p,q}(\mathbb{R}^n)} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Thus, $f \in \mathcal{S}'_0(\mathbb{R}^n)$, which completes the proof of Proposition 3.1. \square

The following lemma is just [1, Lemma 1.2].

Lemma 3.2. *Suppose that $p \in (0, \infty)$, $q \in (0, \infty]$, $\{\mu_k\}_{k \in \mathbb{Z}}$ is a non-negative sequence of complex numbers such that $\{2^k \mu_k\}_{k \in \mathbb{Z}} \in \ell^q$ and φ is a non-negative function having the following property: there exists $\delta \in (0, \min\{1, q/p\})$ such that, for any $k_0 \in \mathbb{N}$, $\varphi \leq \psi_{k_0} + \eta_{k_0}$, where ψ_{k_0} and η_{k_0} are functions, depending on k_0 and satisfying*

$$2^{k_0 p} \left[d_{\psi_{k_0}}(2^{k_0}) \right]^\delta \leq \tilde{C} \sum_{k=-\infty}^{k_0-1} \left[2^k (\mu_k)^\delta \right]^p, \quad 2^{k_0 \delta p} d_{\eta_{k_0}}(2^{k_0}) \leq \tilde{C} \sum_{k=k_0}^{\infty} \left[2^{k \delta} \mu_k \right]^p$$

for some positive constant \tilde{C} independent of k_0 . Then $\varphi \in L^{p,q}(\mathbb{R}^n)$ and

$$\|\varphi\|_{L^{p,q}(\mathbb{R}^n)} \leq C \|\{2^k \mu_k\}_{k \in \mathbb{Z}}\|_{\ell^q},$$

where C is a positive constant independent of φ and $\{\mu_k\}_{k \in \mathbb{Z}}$.

The following lemma is just [15, Lemma 2.3], which is a slight modification of [21, Theorem 11].

Lemma 3.3. *Suppose that A is a dilation on \mathbb{R}^n . Then there exists a collection*

$$\mathcal{Q} := \{Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in I_k\}$$

of open subsets, where I_k is certain index set, such that

- (i) $|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k| = 0$ for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;
- (ii) for any α, β, k, ℓ with $\ell \geq k$, either $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ or $Q_\alpha^\ell \subset Q_\beta^k$;
- (iii) for each (ℓ, β) and each $k < \ell$, there exists a unique α such that $Q_\beta^\ell \subset Q_\alpha^k$;
- (iv) there exist certain negative integer v and positive integer u such that, for all Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$, there exists $x_{Q_\alpha^k} \in Q_\alpha^k$ satisfying that, for all $x \in Q_\alpha^k$,

$$x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}.$$

In what follows, for convenience, we call $\mathcal{Q} := \{Q_\alpha^k\}_{k \in \mathbb{Z}, \alpha \in I_k}$ from Lemma 3.3 *dyadic cubes* and k the *level*, denoted by $\ell(Q_\alpha^k)$, of the dyadic cube Q_α^k with $k \in \mathbb{Z}$ and $\alpha \in I_k$.

Remark 3.4. In the definition of (p, r, s) -atoms (see Definition 2.7(i)), if we replace dilated balls \mathfrak{B} (see (2.5)) by dyadic cubes, from Lemma 3.3, it follows that the corresponding anisotropic atomic Hardy-Lorentz space coincides with the original one (see Definition 2.7(ii)) in the sense of equivalent quasi-norms.

The following Calderón reproducing formula is just [15, Proposition 2.14].

Lemma 3.5. *Let $s \in \mathbb{Z}_+$ and A be a dilation on \mathbb{R}^n . Then there exist $\theta, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that*

- (i) $\text{supp } \theta \subset B_0$, $\int_{\mathbb{R}^n} x^\gamma \theta(x) dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$, $\widehat{\theta}(\xi) \geq C$ for all $\xi \in \{x \in \mathbb{R}^n : a \leq \rho(x) \leq b\}$, where $0 < a < b < 1$ and C are positive constants;
- (ii) $\text{supp } \widehat{\psi}$ is compact and bounded away from the origin;
- (iii) $\sum_{j \in \mathbb{Z}} \widehat{\psi}((A^*)^j \xi) \widehat{\theta}((A^*)^j \xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, where A^* denotes the adjoint of A .

Moreover, for all $f \in \mathcal{S}'_0(\mathbb{R}^n)$, $f = \sum_{j \in \mathbb{Z}} f * \psi_j * \theta_j$ in $\mathcal{S}'(\mathbb{R}^n)$.

Now we prove Theorem 2.11.

Proof of Theorem 2.11. We first prove the necessity of Theorem 2.11. To this end, assume that $f \in H_A^{p,q}(\mathbb{R}^n)$. By Proposition 3.1, $f \in \mathcal{S}'_0(\mathbb{R}^n)$. It remains to show $S(f) \in L^{p,q}(\mathbb{R}^n)$ and $\|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$. We prove this by three steps.

Step 1. By Lemma 2.8 and Remark 3.4, for $f \in H_A^{p,q}(\mathbb{R}^n) = H_A^{p,r,s,q}(\mathbb{R}^n)$, we know that there exists a sequence $\{a_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}}$ of (p, r, s) -atoms, respectively, supported on $\{Q_i^k\}_{i \in \mathbb{N}, k \in \mathbb{Z}} \subset \mathcal{Q}$ such that $f = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k$ in $\mathcal{S}'(\mathbb{R}^n)$, $\lambda_i^k \sim 2^k |Q_i^k|^{1/p}$ for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} \chi_{Q_i^k}(x) \lesssim 1$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, and

$$(3.1) \quad \|f\|_{H_A^{p,q}(\mathbb{R}^n)} \sim \left[\sum_{k \in \mathbb{Z}} \left(\sum_{i \in \mathbb{N}} |\lambda_i^k|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Step 2. In this step, we prove that, for all $j \in \mathbb{Z}_+$ and

$$(3.2) \quad x \in U_j := x_Q + (B_{v[\ell(Q)-j-1]+u+2\tau} \setminus B_{v[\ell(Q)-j]+u+2\tau}),$$

it holds true that

$$(3.3) \quad S(a)(x) \lesssim (1 - vj)^{1/2} b^{\frac{1+\epsilon}{p} vj} b^{-\frac{v\ell(Q)}{r}} \|a\|_{L^r(Q)}$$

for some $\epsilon \in (0, \infty)$, where v, u are as in Lemma 3.3 and a is a (p, r, s) -atom supported on a dyadic cube Q .

To this end, for any $j \in \mathbb{Z}_+$ and $x \in U_j$, we write

$$[S(a)(x)]^2 = \left[\sum_{k > -v\ell(Q)-u} + \sum_{k \leq -v\ell(Q)-u} \right] b^k \int_{x+B_{-k}} |a * \varphi_{-k}(y)|^2 dy =: \text{II}_1 + \text{II}_2.$$

For II_1 , by $-k < v\ell(Q) + u$ and (2.7), for all $y \in x + B_{-k}$, we know that

$$y \in x + B_{-k} \subset x_Q + (B_{v[\ell(Q)-j]+u+2\tau})^{\complement} + B_{-k} \subset x_Q + (B_{v[\ell(Q)-j]+u+\tau})^{\complement},$$

which, together with (2.7), further implies that, for all $z \in Q \subset x_Q + B_{v\ell(Q)+u}$,

$$y - z \in (B_{v[\ell(Q)-j]+u})^{\complement},$$

namely,

$$(3.4) \quad \rho(y - z) \geq b^{v[\ell(Q) - j] + u}.$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, taking some $\tilde{N} \in \mathbb{N}$ to be fixed later, by (3.4), the Hölder inequality and Lemma 3.3(iv), we conclude that, for all $y \in x + B_{-k}$,

$$(3.5) \quad \begin{aligned} |a * \varphi_{-k}(y)| &\lesssim b^k \int_Q |a(z) \varphi(A^k(y - z))| dz \\ &\lesssim b^k \int_Q |a(z)| \frac{1}{\max \left\{ 1, [\rho(A^k(y - z))]^{\tilde{N}+1} \right\}} dz \\ &\lesssim b^{k - (\tilde{N}+1)\{k+v[\ell(Q)-j]+u\}} \int_Q |a(z)| dz \\ &\lesssim b^{k - (\tilde{N}+1)\{k+v[\ell(Q)-j]+u\}} b^{\frac{v\ell(Q)}{r}} \|a\|_{L^r(Q)} \\ &\sim b^{-\tilde{N}\{k+v[\ell(Q)-j]\}} b^{vj} b^{-\frac{v\ell(Q)}{r}} \|a\|_{L^r(Q)}. \end{aligned}$$

By $s \geq \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor$, choosing $\tilde{N} \in \mathbb{N}$ such that $b^{\tilde{N}} > b(\lambda_-)^{s+1} > b^{1/p}$, it follows that there exists a positive constant ϵ such that $b^{\tilde{N}} > b^{\frac{1+\epsilon}{p}}$. This, combined with (3.5), implies that

$$(3.6) \quad \begin{aligned} \text{II}_1 &\lesssim \sum_{k > -v\ell(Q) - u} b^{-2\tilde{N}\{k+v[\ell(Q)-j]\}} b^{2vj} b^{-\frac{2v\ell(Q)}{r}} \|a\|_{L^r(Q)}^2 \\ &\lesssim b^{2\tilde{N}vj} b^{-\frac{2v\ell(Q)}{r}} \|a\|_{L^r(Q)}^2 \\ &\lesssim b^{2\frac{1+\epsilon}{p}vj} b^{-\frac{2v\ell(Q)}{r}} \|a\|_{L^r(Q)}^2 \lesssim (1 - vj) b^{2vj} b^{\frac{1+\epsilon}{p}} b^{-\frac{2v\ell(Q)}{r}} \|a\|_{L^r(Q)}^2, \end{aligned}$$

where the second and the last inequalities follow from the fact $v \in (-\infty, 0)$.

For II_2 , by $A^k(z - x_Q) \in B_{k+v\ell(Q)+u}$ for $z \in Q$, the fact that $k + v\ell(Q) + u \leq 0$ and (2.4), we have

$$(3.7) \quad |A^k(z - x_Q)| \leq (\lambda_-)^{k+v\ell(Q)+u}.$$

Moreover, for all $y \in x + B_{-k}$, if $v[\ell(Q) - j] + u > -k$, by (2.7), we obtain

$$y \in x + B_{-k} \subset x_Q + (B_{v[\ell(Q)-j]+u+2\tau})^{\mathbb{G}} + B_{-k} \subset x_Q + (B_{v[\ell(Q)-j]+u+\tau})^{\mathbb{G}}.$$

Using this, for all $\zeta \in Q \subset x_Q + B_{v\ell(Q)+u}$, by (2.7) again, we further know that $y - \zeta \in (B_{v[\ell(Q)-j]+u})^{\mathbb{G}}$, which implies that

$$\rho(y - \zeta) \geq b^{v[\ell(Q) - j] + u}$$

and hence

$$(3.8) \quad \sup_{\zeta \in Q} \frac{1}{\max \left\{ 1, [\rho(A^k(y - \zeta))]^{N+1} \right\}} \leq \min \left\{ 1, b^{-(N+1)\{k+v[\ell(Q)-j]+u\}} \right\}.$$

If $v[\ell(Q) - j] + u \leq -k$, since $b^{-(N+1)\{k+v[\ell(Q)-j]+u\}} \geq 1$, obviously, (3.8) still holds true.

Noticing that a has the vanishing moments up to order s , by Taylor's remainder theorem and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we conclude that, for all $y \in x + B_{-k}$,

$$\begin{aligned}
 (3.9) \quad & |a * \varphi_{-k}(y)| \\
 &= b^k \left| \int_Q a(z) \left\{ \varphi(A^k(y-z)) - \sum_{|\gamma| \leq s} \frac{\partial^\gamma \varphi(A^k(y-x_Q))}{\gamma!} [A^k(x_Q-z)]^\gamma \right\} dz \right| \\
 &\lesssim b^k \int_Q |a(z)| |A^k(x_Q-z)|^{s+1} \sup_{\zeta \in Q} \frac{1}{\max \{1, \rho(A^k(y-\zeta))^{\tilde{N}+1}\}} dz.
 \end{aligned}$$

Since $b^{\tilde{N}} > b(\lambda_-)^{s+1} > b^{\frac{1+\epsilon}{p}}$, combining (3.7), (3.8) and (3.9), by the Hölder inequality and Lemma 3.3(iv), we have

$$\begin{aligned}
 (3.10) \quad \Pi_2 &= \left[\sum_{-v[\ell(Q)-j]-u < k \leq -v\ell(Q)-u} + \sum_{k \leq -v[\ell(Q)-j]-u} \right] b^k \int_{x+B_{-k}} |a * \varphi_{-k}(y)|^2 dy \\
 &\lesssim \left\{ \sum_{-v[\ell(Q)-j]-u < k \leq -v\ell(Q)-u} \left[(\lambda_-)^{(s+1)[k+v\ell(Q)+u]} \right. \right. \\
 &\quad \times b^{-(\tilde{N}+1)\{k+v[\ell(Q)-j]+u\}} b^{k+\frac{v\ell(Q)}{r}} \left. \right]^2 \\
 &\quad + \sum_{k \leq -v[\ell(Q)-j]-u} \left[(\lambda_-)^{(s+1)[k+v\ell(Q)+u]} b^{k+\frac{v\ell(Q)}{r}} \right]^2 \left. \right\} \|a\|_{L^r(Q)}^2 \\
 &\lesssim \left\{ \sum_{-v[\ell(Q)-j]-u < k \leq -v\ell(Q)-u} b^{-\frac{2v\ell(Q)}{r}} [b(\lambda_-)^{s+1}]^{2vj} \right. \\
 &\quad + \sum_{k \leq -v[\ell(Q)-j]-u} b^{-\frac{2v\ell(Q)}{r}} [b(\lambda_-)^{s+1}]^{2vj} \left. \right\} \|a\|_{L^r(Q)}^2 \\
 &\lesssim (1-vj) b^{2vj} b^{\frac{1+\epsilon}{p}} b^{-\frac{2v\ell(Q)}{r}} \|a\|_{L^r(Q)}^2.
 \end{aligned}$$

Combining the estimates of Π_1 (see (3.6)) and Π_2 (see (3.10)), we know that (3.3) holds true.

Step 3. In this step, we use (3.3) to show $S(f) \in L^{p,q}(\mathbb{R}^n)$ and $\|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$.

To this end, it suffices only to consider $N = N_{(p)} := \lfloor (\frac{1}{p} - 1) \frac{\ln b}{\ln \lambda_-} \rfloor + 2$. For all $k \in \mathbb{Z}$, let $\mu_k := (\sum_{i \in \mathbb{N}} |Q_i^k|)^{1/p}$. Clearly, for $r \in (1, \infty]$, there exists $\delta \in (0, 1)$ such that $\max\{\frac{1}{r}, \frac{1}{1+\epsilon}\} < \delta < 1$ and $\delta p < 1$, where ϵ is as in (3.3). Notice that, for any fixed $k_0 \in \mathbb{Z}$

and all $x \in \mathbb{R}^n$,

$$S(f)(x) \leq S\left(\sum_{k=-\infty}^{k_0-1} \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k\right)(x) + \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k| S(a_i^k)(x) =: \psi_{k_0}(x) + \eta_{k_0}(x).$$

To obtain the desired conclusion, we consider two cases: $q/p \in [1, \infty]$ and $q/p \in (0, 1)$.

Case 1: $q/p \in [1, \infty]$. For this case, if we prove that

$$(3.11) \quad 2^{k_0 p} \left[d_{\psi_{k_0}}(2^{k_0}) \right]^\delta \lesssim \sum_{k=-\infty}^{k_0-1} \left[2^k \mu_k^\delta \right]^p \quad \text{and} \quad 2^{k_0 \delta p} d_{\eta_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} \left[2^{k \delta} \mu_k \right]^p,$$

then, noticing that $\delta \in (0, q/p)$, by Lemma 3.2, the fact that $|Q_i^k| \sim \frac{|\lambda_i^k|^p}{2^{kp}}$ and (3.1), we find that

$$\|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \left\| \left\{ 2^k \mu_k \right\}_{k \in \mathbb{Z}} \right\|_{\ell^q} \lesssim \left[\sum_{k \in \mathbb{Z}} \left(\sum_{i \in \mathbb{N}} |\lambda_i^k|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \sim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$$

with the usual interpretation for $q = \infty$, which implies that $S(f) \in L^{p,q}(\mathbb{R}^n)$ and

$$\|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$$

as desired.

Now we show (3.11). To this end, we first estimate ψ_{k_0} . Notice that a_i^k is a (p, r, s) -atom, $\text{supp } a \subset Q_i^k$, $\sum_{i \in \mathbb{N}} \chi_{Q_i^k} \lesssim 1$ and $\lambda_i^k \sim 2^k |Q_i^k|^{1/p}$. For $r \in (1, \infty)$, by the Hölder inequality, we conclude that, for $\sigma := 1 - \frac{p}{r\delta} > 0$ and all $x \in \mathbb{R}^n$,

$$\begin{aligned} \psi_{k_0}(x) &\leq \sum_{k=-\infty}^{k_0-1} S\left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k\right)(x) \\ &\leq \left(\sum_{k=-\infty}^{k_0-1} 2^{k\sigma r'}\right)^{1/r'} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r} \left[S\left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k\right)(x) \right]^r \right\}^{1/r} \\ &= C_5 2^{k_0 \sigma} \left\{ \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r} \left[S\left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k\right)(x) \right]^r \right\}^{1/r}, \end{aligned}$$

where $C_5 := (\frac{1}{2^{\sigma r'} - 1})^{1/r'}$, which, together with [15, Theorem 3.2], further implies that

$$(3.12) \quad \begin{aligned} &2^{k_0 p} \left[d_{\psi_{k_0}}(2^{k_0}) \right]^\delta \\ &\leq 2^{k_0 p} \left| \left\{ x \in \mathbb{R}^n : C_5^r \sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r} \left[S\left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k\right)(x) \right]^r > 2^{k_0 r(1-\sigma)} \right\} \right|^\delta \end{aligned}$$

$$\begin{aligned}
&\leq C_5^{\delta r} 2^{k_0 p} 2^{-k_0 r \delta (1-\sigma)} \left\{ \int_{\mathbb{R}^n} \sum_{k=-\infty}^{k_0-1} 2^{-k \sigma r} \left[S \left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k \right) (x) \right]^r dx \right\}^{\delta} \\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k \sigma r} \int_{\mathbb{R}^n} \left| \sum_{i \in \mathbb{N}} \lambda_i^k a_i^k(x) \right|^r dx \right]^{\delta} \\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k \sigma r} \sum_{i \in \mathbb{N}} |\lambda_i^k|^r \int_{Q_i^k} |a_i^k(x)|^r dx \right]^{\delta} \\
&\lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k \sigma r} \sum_{i \in \mathbb{N}} 2^{kr} \left| Q_i^k \right|^{\frac{r}{p}} \left| Q_i^k \right|^{(\frac{1}{r}-\frac{1}{p})r} \right]^{\delta} \\
&\lesssim \sum_{k=-\infty}^{k_0-1} 2^{kp} \left(\sum_{i \in \mathbb{N}} \left| Q_i^k \right| \right)^{\delta} \sim \sum_{k=-\infty}^{k_0-1} \left[2^k \mu_k^{\delta} \right]^p,
\end{aligned}$$

which is the desired estimate of ψ_{k_0} for $r \in (1, \infty)$ in (3.11).

For $r = \infty$, by [15, Theorem 3.2] again, we know that

$$\begin{aligned}
(3.13) \quad 2^{k_0 p} \left[d_{\psi_{k_0}}(2^{k_0}) \right]^{\delta} &= 2^{k_0 p} \left| \left\{ x \in \mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0} \right\} \right|^{\delta} \\
&\leq 2^{k_0(p-\delta\tilde{r})} \left\{ \sum_{k=-\infty}^{k_0-1} \int_{\mathbb{R}^n} \left[S \left(\sum_{i \in \mathbb{N}} \lambda_i^k a_i^k \right) (x) \right]^{\tilde{r}} dx \right\}^{\delta} \\
&\lesssim 2^{k_0(p-\delta\tilde{r})} \left\{ \sum_{k=-\infty}^{k_0-1} \sum_{i \in \mathbb{N}} \int_{x_i^k + B_{\ell_i^k}} \left| \lambda_i^k a_i^k \right|^{\tilde{r}}(x) dx \right\}^{\delta} \\
&\lesssim \sum_{k=-\infty}^{k_0-1} 2^{kp} \left(\sum_{i \in \mathbb{N}} \left| Q_i^k \right| \right)^{\delta} \sim \sum_{k=-\infty}^{k_0-1} \left[2^k \mu_k^{\delta} \right]^p,
\end{aligned}$$

where $\tilde{r} \in (1, \infty)$ satisfies that $\delta\tilde{r} > p$, which, combined with (3.12), implies the desired estimate of ψ_{k_0} in (3.11).

In order to estimate η_{k_0} , we claim that, for all $i \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$(3.14) \quad \int_{\mathbb{R}^n} \left[S(a_i^k)(x) \right]^{\delta p} dx \lesssim \left| Q_i^k \right|^{1-\delta}.$$

Assume that (3.14) holds true for the moment. Then, by (3.14), we have

$$\begin{aligned}
(3.15) \quad 2^{k_0 \delta p} d_{\eta_{k_0}}(2^{k_0}) &= 2^{k_0 \delta p} \left| \left\{ x \in \mathbb{R}^n : \left[\sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k| S(a_i^k) \right]^{\delta p} (x) > 2^{k_0 \delta p} \right\} \right| \\
&\leq \int_{\mathbb{R}^n} \left[\sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k| S(a_i^k) \right]^{\delta p} (x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k|^{\delta p} \int_{\mathbb{R}^n} [S(a_i^k)(x)]^{\delta p} dx \\
&\lesssim \sum_{k=k_0}^{\infty} \sum_{i \in \mathbb{N}} |\lambda_i^k|^{\delta p} |Q_i^k|^{1-\delta} \lesssim \sum_{k=k_0}^{\infty} 2^{k\delta p} \sum_{i \in \mathbb{N}} |Q_i^k| \sim \sum_{k=k_0}^{\infty} [2^{k\delta} \mu_k]^p,
\end{aligned}$$

which is the desired estimate of η_{k_0} in (3.11).

Now we show (3.14). To this end, we write

$$\begin{aligned}
\int_{\mathbb{R}^n} [S(a)(x)]^{\delta p} dx &= \int_{x_Q + B_{v\ell(Q)+u+2\tau}} [S(a)(x)]^{\delta p} dx + \int_{x_Q + (B_{v\ell(Q)+u+2\tau})^c} \cdots \\
&=: I_1 + I_2,
\end{aligned}$$

where a is a (p, r, s) -atom supported on the dyadic cube Q and v, u are as in Lemma 3.3. By the Hölder inequality, [15, Theorem 3.2] again and Lemma 3.3(iv), we know that

$$\begin{aligned}
(3.16) \quad I_1 &\leq \left\{ \int_{x_Q + B_{v\ell(Q)+u+2\tau}} [S(a)(x)]^r dx \right\}^{\delta p/r} |B_{v\ell(Q)+u+2\tau}|^{1-\delta p/r} \\
&\lesssim \|a\|_{L^r(\mathbb{R}^n)}^{\delta p} |Q|^{1-\delta p/r} \lesssim |Q|^{\delta p(1/r-1/p)} |Q|^{1-\delta p/r} \sim |Q|^{1-\delta}.
\end{aligned}$$

Noticing that $(1+\epsilon)\delta > 1$, by (3.3) and Lemma 3.3(iv) again, we find that

$$\begin{aligned}
I_2 &= \sum_{j=0}^{\infty} \int_{U_j} [S(a)(x)]^{\delta p} dx \\
&\lesssim \sum_{j=0}^{\infty} (1-vj)^{\delta p/2} b^{v\ell(Q)-vj} b^{(1+\epsilon)\delta vj} b^{-\delta p \frac{v\ell(Q)}{r}} \|a\|_{L^r(\mathbb{R}^n)}^{\delta p} \\
&\lesssim \sum_{j=0}^{\infty} (1-vj)^{\delta p/2} b^{v\ell(Q)-vj} b^{(1+\epsilon)\delta vj} b^{-\delta p \frac{v\ell(Q)}{r}} b^{v\ell(Q)\delta p(\frac{1}{r}-\frac{1}{p})} \\
&\sim \sum_{j=0}^{\infty} b^{v\ell(Q)(1-\delta)} (1-vj)^{\delta p/2} b^{[(1+\epsilon)\delta-1]vj} \sim |Q|^{1-\delta},
\end{aligned}$$

where U_j is as in (3.2), which, together with (3.16), implies that (3.14) holds true. This finishes the proof of the case when $q/p \in [1, \infty]$.

Case 2: $q/p \in (0, 1)$. In this case, when $r \in (1, \infty)$, similar to (3.12), we have

$$(3.17) \quad 2^{k_0 p} [d_{\psi_{k_0}}(2^{k_0})]^{\delta} \lesssim \left[\sum_{k=-\infty}^{k_0-1} 2^{-k\sigma r} \sum_{i \in \mathbb{N}} 2^{kr} |Q_i^k|^{\frac{r}{p}} |Q_i^k|^{(\frac{1}{r}-\frac{1}{p})r} \right]^{\delta} \sim \left(\sum_{k=-\infty}^{k_0-1} 2^{\frac{kp}{\delta}} \mu_k^p \right)^{\delta}.$$

By some calculations similar to (3.13), we easily know that (3.17) also holds true for $r = \infty$. This further implies that

$$(3.18) \quad \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \left| \left\{ x \in \mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0} \right\} \right|^{\frac{q}{p}} \lesssim \sum_{k_0 \in \mathbb{Z}} 2^{k_0(q-\frac{q}{\delta})} \sum_{k=-\infty}^{k_0-1} 2^{\frac{kq}{\delta}} \mu_k^q$$

$$\sim \sum_{k \in \mathbb{Z}} \sum_{k_0=k+1}^{\infty} 2^{k_0(q-\frac{q}{\delta})} 2^{\frac{kq}{\delta}} \mu_k^q \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} \mu_k^q.$$

On the other hand, similar to (3.15), we deduce that

$$2^{k_0 \delta p} d_{\eta_{k_0}}(2^{k_0}) \lesssim \sum_{k=k_0}^{\infty} \left[2^{k \delta} \mu_k \right]^p,$$

which implies that

$$\begin{aligned} & 2^{k_0 \delta p} \left| \left\{ x \in \mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0} \right\} \right| \\ & \lesssim \sum_{k=k_0}^{\infty} 2^{-k \tilde{\delta} p} \left[2^{k(1-\tilde{\delta})} \mu_k \right]^p \lesssim 2^{-k_0 \tilde{\delta} p} \left\{ \sum_{k=k_0}^{\infty} \left[2^{k(1-\tilde{\delta})} \mu_k \right]^q \right\}^{\frac{p}{q}}, \end{aligned}$$

where $\tilde{\delta} := \frac{1-\delta}{2}$. Therefore,

$$\begin{aligned} (3.19) \quad \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \left| \left\{ x \in \mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0} \right\} \right|^{\frac{q}{p}} & \lesssim \sum_{k_0 \in \mathbb{Z}} 2^{k_0 \tilde{\delta} q} \sum_{k=k_0}^{\infty} \left[2^{k(1-\tilde{\delta})} \mu_k \right]^q \\ & \sim \sum_{k \in \mathbb{Z}} \left[2^{k(1-\tilde{\delta})} \mu_k \right]^q \sum_{k_0=-\infty}^k 2^{k_0 \tilde{\delta} q} \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} \mu_k^q. \end{aligned}$$

Notice that $\mu_k := (\sum_{i \in \mathbb{N}} |Q_i^k|)^{1/p}$ and $\lambda_i^k \sim 2^k |Q_i^k|^{1/p}$. Combining (2.1), (3.18), (3.19) and (3.1), we further conclude that

$$\begin{aligned} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q & \sim \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^{k_0} \right\} \right|^{\frac{q}{p}} \\ & \lesssim \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \left| \left\{ x \in \mathbb{R}^n : \psi_{k_0}(x) > 2^{k_0} \right\} \right|^{\frac{q}{p}} \\ & \quad + \sum_{k_0 \in \mathbb{Z}} 2^{k_0 q} \left| \left\{ x \in \mathbb{R}^n : \eta_{k_0}(x) > 2^{k_0} \right\} \right|^{\frac{q}{p}} \\ & \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} \mu_k^q \sim \sum_{k \in \mathbb{Z}} \left[\sum_{i \in \mathbb{N}} |\lambda_i^k|^p \right]^{\frac{q}{p}} \sim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}^q, \end{aligned}$$

which implies that $S(f) \in L^{p,q}(\mathbb{R}^n)$ and $\|S(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p,q}(\mathbb{R}^n)}$. This finishes the proof of Case 2 and hence the proof of the necessity of Theorem 2.11.

Now we show the sufficiency of Theorem 2.11, namely, to show that, if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{p,q}(\mathbb{R}^n)$, then $f \in H_A^{p,q}(\mathbb{R}^n)$ and

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

We prove this by six steps.

Step (i). For each $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : S(f)(x) > 2^k\}$ and

$$\mathcal{Q}_k := \left\{ Q \in \mathcal{Q} : |Q \cap \Omega_k| > \frac{|Q|}{2} \text{ and } |Q \cap \Omega_{k+1}| \leq \frac{|Q|}{2} \right\}.$$

Obviously, for any $Q \in \mathcal{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{Q}_k$. We denote the set of all *maximal dyadic cubes* in \mathcal{Q}_k by $\{Q_i^k\}_i$, namely, there exist no $Q \in \mathcal{Q}_k$ such that $Q_i^k \subsetneq Q$ for any i .

For any $Q \in \mathcal{Q}$, let

$$\widehat{Q} := \{(y, t) \in \mathbb{R}^n \times \mathbb{R} : y \in Q, t \sim v\ell(Q) + u\},$$

here and hereafter, $t \sim v\ell(Q) + u$ always means

$$(3.20) \quad v\ell(Q) + u + \tau \leq t < v[\ell(Q) - 1] + u + \tau,$$

where u, v are as in Lemma 3.3 and $\ell(Q)$ is the level of Q . Observe that, in the above inequality, v is negative. Clearly, $\{\widehat{Q}\}_{Q \in \mathcal{Q}}$ are mutually disjoint and

$$(3.21) \quad \mathbb{R}^n \times \mathbb{R} = \bigcup_{k \in \mathbb{Z}} \bigcup_i B_{k,i},$$

where, for any $k \in \mathbb{Z}$ and i , let $B_{k,i} := \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \widehat{Q}$. It is easy to see that $\{B_{k,i}\}_{k \in \mathbb{Z}, i}$ are mutually disjoint by Lemma 3.3(ii).

Let ψ and θ be as in Lemma 3.5. Then θ has the vanishing moments up to order s with $s \geq \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor$. By Lemma 3.5, the properties of the tempered distributions (see [39, Theorem 2.3.20] or [67, Theorem 3.13]) and (3.21), we find that, for all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ with $S(f) \in L^{p,q}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} f * \psi_k * \theta_k(x) = \int_{\mathbb{R}^n \times \mathbb{R}} f * \psi_t(y) * \theta_t(x - y) dy dm(t) \\ &= \sum_{k \in \mathbb{Z}} \sum_i \int_{B_{k,i}} f * \psi_t(y) * \theta_t(x - y) dy dm(t) =: \sum_{k \in \mathbb{Z}} \sum_i h_i^k(x) \end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where, for each $k \in \mathbb{Z}$, i and $x \in \mathbb{R}^n$,

$$\begin{aligned} (3.22) \quad h_i^k(x) &:= \int_{B_{k,i}} f * \psi_t(y) * \theta_t(x - y) dy dm(t) \\ &= \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\widehat{Q}} f * \psi_t(y) * \theta_t(x - y) dy dm(t) =: \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q(x) \end{aligned}$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$, and $m(t)$ is the *counting measure* on \mathbb{R} .

Step (ii). In this step, we prove that, for all $x \in \mathbb{R}^n$,

$$(3.23) \quad \left[S \left(\sum_{Q \in \mathcal{R}} e_Q \right) (x) \right]^2 \lesssim \sum_{Q \in \mathcal{R}} [M_{\text{HL}}(c_Q \chi_Q)(x)]^2,$$

where \mathcal{R} is any set of dyadic cubes in \mathbb{R}^n , e_Q is as in (3.22) and, for each $Q \in \mathcal{R}$,

$$c_Q := \left[\int_{\widehat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{b^t} \right]^{1/2}.$$

Indeed, we can prove (3.23) via borrowing some ideas from the proof of [15, Lemma 4.7]. For each $Q \in \mathcal{Q}$, let $\widetilde{Q} := c_Q + B_{v[\ell(Q)-1]+u+2\tau}$, where c_Q is as in (3.23). By repeating the proof of [15, Lemma 4.7, pp. 412-413], we find that, for all $x \in \mathbb{R}^n$,

$$\left[S \left(\sum_{Q \in \mathcal{R}} e_Q \right) (x) \right]^2 \lesssim \sum_{Q \in \mathcal{R}} (c_Q)^2 [M_{\text{HL}}(\chi_Q)(x)]^2 \left\{ \sum_{R \in \mathcal{Q}, \widetilde{Q} \cap \widetilde{R} \neq \emptyset} b^{(s+1)v|\ell(Q)-\ell(R)| \frac{\ln(\lambda_-)}{\ln b}} \right\}.$$

By this and the following estimate from [46, p. 295]:

$$\sum_{R \in \mathcal{Q}, \widetilde{Q} \cap \widetilde{R} \neq \emptyset} b^{(s+1)v|\ell(Q)-\ell(R)| \frac{\ln(\lambda_-)}{\ln b}} \lesssim 1$$

(see also [15, (4.18)] for the case of two parameters), we obtain (3.23).

Next we show that, for each $k \in \mathbb{Z}$ and i , h_i^k is a multiple of a (p, r, s) -atom. This is completed by Step (iii) through (v) below.

Step (iii). For all $x \in \text{supp } h_i^k$, by (3.22), $h_i^k(x) \neq 0$ implies that there exists $Q \subset Q_i^k$ and $Q \in \mathcal{Q}_k$ such that $e_Q(x) \neq 0$. Then there exists $(y, t) \in \widehat{Q}$ such that $A^{-t}(x - y) \in B_0$. By this, Lemma 3.3(iv), (3.20) and (2.6), we have

$$x \in y + B_t \subset x_Q + B_{v\ell(Q)+u} + B_{v[\ell(Q)-1]+u+2\tau} \subset x_Q + B_{v[\ell(Q)-1]+u+2\tau}.$$

Therefore,

$$\text{supp } e_Q \subset x_Q + B_{v[\ell(Q)-1]+u+2\tau}.$$

By this, $h_i^k = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$, (ii) and (iv) of Lemma 3.3 and (2.6), we further conclude that

$$\begin{aligned} (3.24) \quad \text{supp } h_i^k &\subset \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} x_Q + B_{v[\ell(Q)-1]+u+2\tau} \\ &\subset x_{Q_i^k} + B_{v\ell(Q_i^k)+u} + B_{v[\ell(Q_i^k)-1]+u+2\tau} \\ &\subset x_{Q_i^k} + B_{v[\ell(Q_i^k)-1]+u+3\tau} =: B_i^k. \end{aligned}$$

Step (iv). Notice that, for each $Q \in \mathcal{Q}_k$ and $x \in Q$, by Lemma 3.3(iv), we know that

$$M_{\text{HL}}(\chi_{\Omega_k})(x) \geq \frac{1}{b^{v\ell(Q)+u}} \int_{x_Q + B_{v\ell(Q)+u}} \chi_{\Omega_k}(z) dz > b^{-2u} \frac{|\Omega_k \cap Q|}{|Q|} > \frac{1}{2} b^{-2u},$$

which further implies that

$$(3.25) \quad \bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} Q \subset \widehat{\Omega}_k := \left\{ x \in \mathbb{R}^n : M_{\text{HL}}(\chi_{\Omega_k})(x) > \frac{1}{2} b^{-2u} \right\}.$$

Moreover, for all $Q \in \mathcal{Q}_k$ and $x \in Q$, by Lemma 3.3(iv) and $Q \subset \widehat{\Omega}_k$, we obtain

$$M_{\text{HL}} \left(\chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right) (x) \gtrsim \frac{1}{|Q|} \int_Q \chi_{\widehat{\Omega}_k \setminus \Omega_{k+1}}(z) dz \gtrsim \frac{|Q| - |Q|/2}{|Q|} \gtrsim \frac{\chi_Q(x)}{2}.$$

By this, [15, Theorem 3.2], (3.23) and [12, Theorem 2.5], we find that, for $r \in (1, \infty)$,

$$\begin{aligned} (3.26) \quad \left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q \right\|_{L^r(\mathbb{R}^n)} &\lesssim \left\| S \left(\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q \right) \right\|_{L^r(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} [M_{\text{HL}}(c_Q \chi_Q)]^2 \right\}^{1/2} \right\|_{L^r(\mathbb{R}^n)} \\ &\lesssim \left\| \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \chi_Q \right]^{1/2} \right\|_{L^r(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \left[M_{\text{HL}} \left(c_Q \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right) \right]^2 \right\}^{1/2} \right\|_{L^r(\mathbb{R}^n)} \\ &\lesssim \left\| \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right]^{1/2} \right\|_{L^r(\mathbb{R}^n)}. \end{aligned}$$

On the other hand, for any $Q \in \mathcal{Q}_k$, $x \in Q$ and $(y, t) \in \widehat{Q}$, by Lemma 3.3(iv), (2.6) and (3.20), we have

$$x - y \in B_{v\ell(Q)+u} + B_{v\ell(Q)+u} \subset B_{v\ell(Q)+u+\tau} \subset B_t,$$

which, combined with the disjointness of $\{\widehat{Q}\}_{Q \subset Q_i^k}$, further implies that

$$\begin{aligned} (3.27) \quad &\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \\ &= \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{b^t} \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \\ &\lesssim [S(f)(x)]^2 \chi_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x). \end{aligned}$$

By the definition of $\widehat{\Omega}_k$ (see (3.25)), we know that, for all $r \in (1, \infty)$,

$$|\widehat{\Omega}_k| \leq (2b^{2u})^r \int_{\mathbb{R}^n} [M_{\text{HL}}(\chi_{\Omega_k})(x)]^r dx \lesssim |\Omega_k|,$$

which, together with (3.27), implies that

$$\begin{aligned}
 (3.28) \quad & \left\| \left\{ \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} (c_Q)^2 \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right\}^{\frac{1}{2}} \right\|_{L^r(\mathbb{R}^n)}^r \\
 & \leq \int_{\mathbb{R}^n} \left[\chi_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \int_{\bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \widehat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{b^t} \right]^{r/2} dx \\
 & \lesssim 2^{kr} |\widehat{\Omega}_k| \lesssim 2^{kr} |\Omega_k| < \infty.
 \end{aligned}$$

For any $N \in \mathbb{N}$, let $\mathcal{Q}_{k,N} := \{Q \in \mathcal{Q}_k : |\ell(Q)| > N\}$. Notice that, if we replace $\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$ by $\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q$ in (3.26), then

$$\begin{aligned}
 & \left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q \right\|_{L^r(\mathbb{R}^n)}^r \\
 & \lesssim \left\| \left[\sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} (c_Q)^2 \chi_{Q \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})} \right]^{1/2} \right\|_{L^r(\mathbb{R}^n)}^r \\
 & \lesssim \int_{\mathbb{R}^n} \chi_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) \left[\int_{\bigcup_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} \widehat{Q}} |\psi_t * f(y)|^2 dy \frac{dm(t)}{b^t} \right]^{r/2} dx.
 \end{aligned}$$

Thus, by (3.28) and the Lebesgue dominated convergence theorem, we conclude that

$$\left\| \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_{k,N}} e_Q \right\|_{L^r(\mathbb{R}^n)} \rightarrow 0$$

as $N \rightarrow \infty$, which implies that $h_i^k = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} e_Q$ in $L^r(\mathbb{R}^n)$. By this, combined with (3.26), (3.27), the definition of B_i^k (see (3.24)) and Lemma 3.3(iv), we further find that

$$\begin{aligned}
 (3.29) \quad \|h_i^k\|_{L^r(\mathbb{R}^n)} & \lesssim \left\{ \int_{\mathbb{R}^n} [S(f)(x)]^r \chi_{Q_i^k \cap (\widehat{\Omega}_k \setminus \Omega_{k+1})}(x) dx \right\}^{1/r} \\
 & \lesssim 2^k |Q_i^k|^{1/r} \leq C_6 2^k |B_i^k|^{1/r},
 \end{aligned}$$

where C_6 is a positive constant independent of f .

Step (v). Recall that θ has the vanishing moments up to $s \geq \lfloor (1/p - 1) \ln b / \ln \lambda_- \rfloor$ and so is e_Q . For all $k \in \mathbb{Z}$, i , $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$ and $x \in \mathbb{R}^n$, let $g(x) := x^\gamma \chi_{B_i^k}(x)$ with B_i^k being as in (3.24), and $r' \in (1, \infty)$ such that $1/r + 1/r' = 1$. Clearly, $g \in L^{r'}(\mathbb{R}^n)$. Therefore, by the fact that $(L^{r'}(\mathbb{R}^n))^* = L^r(\mathbb{R}^n)$, (3.24) and

$$\text{supp } e_Q \subset x_Q + B_{v[\ell(Q)-1]+u+2\tau} \subset B_i^k,$$

we further have

$$\begin{aligned}
 (3.30) \quad \int_{\mathbb{R}^n} h_i^k(x) x^\gamma dx &= \langle h_i^k, g \rangle = \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \langle e_Q, g \rangle \\
 &= \sum_{Q \subset Q_i^k, Q \in \mathcal{Q}_k} \int_{\mathbb{R}^n} e_Q(x) x^\gamma dx = 0.
 \end{aligned}$$

Namely, h_i^k has the vanishing moments up to s , which, together with (3.24) and (3.29), implies that h_i^k is a (p, r, s) -atom modulus a constant supported on B_i^k .

Step (vi). For all $k \in \mathbb{Z}$ and i , let $\lambda_i^k := C_6 2^k |B_i^k|^{1/p}$ and $a_i^k := (\lambda_i^k)^{-1} h_i^k$, where C_6 is a positive constant as in (3.29). Then we have

$$f = \sum_{k \in \mathbb{Z}} \sum_i h_i^k = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

By (3.24) and (3.30), we find that $\text{supp } a_i^k \subset B_i^k$ and a_i^k also has the vanishing moments up to s . By (3.29) and Lemma 3.3(iv), we conclude that $\|a_i^k\|_{L^r(\mathbb{R}^n)} \leq |B_i^k|^{1/r-1/p}$. Thus, a_i^k is a (p, r, s) -atom for all $k \in \mathbb{Z}$ and i .

By the mutual disjointness of $\{Q_i^k\}_{k \in \mathbb{Z}, i}$, Lemma 3.3(iv) again and (2.1), we know that

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}} \left(\sum_i |\lambda_i^k|^p \right)^{\frac{q}{p}} &\sim \sum_{k \in \mathbb{Z}} \left(\sum_i 2^{kp} |B_i^k| \right)^{\frac{q}{p}} \\
 &\sim \sum_{k \in \mathbb{Z}} 2^{kq} \left(\sum_i |Q_i^k| \right)^{\frac{q}{p}} \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} |\Omega_k|^{\frac{q}{p}} \sim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q,
 \end{aligned}$$

which implies that $f \in H_A^{p,q}(\mathbb{R}^n)$ and $\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}$. This finishes the proof of Theorem 2.11. \square

4 Proofs of Theorems 2.12 and 2.13

In this section, via first establishing an anisotropic Fefferman-Stein vector-valued inequality in $L^{p,q}(\mathbb{R}^n)$ and the Calderón reproducing formula, we give out the proofs of Theorems 2.12 and 2.13.

We begin with recalling some notation and establishing several technical lemmas. Suppose that A is a dilation on \mathbb{R}^n . For each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, define $Q_{j,k} := A^{-j}([0, 1)^n + k)$, $\mathcal{Q}_j := \{Q_{j,k} : k \in \mathbb{Z}^n\}$ and $\widehat{\mathcal{Q}} := \bigcup_{j \in \mathbb{Z}} \mathcal{Q}_j$. Recall that $Q_{j,k}$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ is called a *dilated cube* (see, for example, [12, p.1475]). Clearly, for any $k_1, k_2 \in \mathbb{Z}^n$ with $k_1 \neq k_2$, $|Q_{j,k_1} \cap Q_{j,k_2}| = 0$.

Throughout this article, for each $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$B_\rho(x, r) := \{y \in \mathbb{R}^n : \rho(x - y) < r\}.$$

For each dilated cube $Q_{j,k}$, denote its *center* by $c_{Q_{j,k}}$ and its *lower-left corner* $A^{-j}k$ by $x_{Q_{j,k}}$. Via [11, Lemma 2.9(a)], we know that there exists a positive integer $j_0 := j_{(A,n)}$, only depending on A and n , such that, for each $x \in Q_{j,k}$,

$$(4.1) \quad B_\rho(c_{Q_{j,k}}, b^{-j_0-j}) \subset Q_{j,k} \subset B_\rho(x, b^{j_0-j}).$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$(4.2) \quad \varphi_Q(x) := |\det A|^{j/2} \varphi(A^j x - k) = |Q|^{1/2} \varphi_{-j}(x - x_Q),$$

where $Q := Q_{j,k} \in \widehat{\mathcal{Q}}$ and $x_Q := x_{Q_{j,k}}$.

Observe that (\mathbb{R}^n, ρ, dx) is a space of homogeneous type in the sense of Coifman and Weiss [22, 23]. From this and [40, Theorem 1.2], we deduce the following lemma, which is an anisotropic version of [32, Theorem 1].

Lemma 4.1. *Let $r \in (1, \infty]$ and M_{HL} be the Hardy-Littlewood maximal function defined by (2.11).*

- (i) *If $p \in (1, \infty)$, then there exists a positive constant C_7 such that, for all sequences $\{f_k\}_k$ of measurable functions,*

$$\left\| \left\{ \sum_k [M_{\text{HL}}(f_k)]^r \right\}^{1/r} \right\|_{L^p(\mathbb{R}^n)} \leq C_7 \left\| \left[\sum_k |f_k|^r \right]^{1/r} \right\|_{L^p(\mathbb{R}^n)};$$

- (ii) *It holds true that there exists a positive constant C_8 such that, for all sequences $\{f_k\}_k$ of measurable functions,*

$$\left\| \left\{ \sum_k [M_{\text{HL}}(f_k)]^r \right\}^{1/r} \right\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C_8 \left\| \left[\sum_k |f_k|^r \right]^{1/r} \right\|_{L^1(\mathbb{R}^n)}.$$

The following lemma is just [46, Lemma 3.3].

Lemma 4.2. *Suppose that $\ell, M \in \mathbb{N}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ which have the vanishing moments up to order ℓ . Then there exists a positive constant $C_{(\ell,M)}$, depending on ℓ and M , such that, for any $i, j \in \mathbb{Z}$ with $i \geq j$ and $x \in \mathbb{R}^n$,*

$$|\varphi_{-i} * \psi_{-j}(x)| \leq C_{(\ell,M)} b^{j-(i-j)(\ell+1)\zeta_-} [1 + \rho(A^j x)]^{-M},$$

where $\zeta_- := \ln(\lambda_-)/\ln b$.

The following discrete Calderón reproducing formula is just [46, Lemma 3.2], which is an anisotropic version of [37, Theorem 6.16].

Lemma 4.3. *Suppose that $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that $\text{supp } \widehat{\Psi}, \text{supp } \widehat{\Phi} \subset [-1, 1]^n \setminus \{\vec{0}_n\}$ and, for all $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,*

$$\sum_{j \in \mathbb{Z}} \overline{\widehat{\Phi}((A^*)^j \xi)} \widehat{\Psi}((A^*)^j \xi) = 1,$$

where A^* denotes the transpose of A . Then, for any $f \in \mathcal{S}'_0(\mathbb{R}^n)$,

$$f(\cdot) = \sum_{Q \in \widehat{\mathcal{Q}}} \langle f, \Phi_Q \rangle \Psi_Q(\cdot) = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_j} b^j f * \widetilde{\Phi}_j(x_Q) \Psi_j(\cdot - x_Q) \text{ holds true in } \mathcal{S}'(\mathbb{R}^n),$$

where $\widetilde{\Phi}(\cdot) := \overline{\Phi(-\cdot)}$, and Φ_Q, Ψ_Q are defined as in (4.2).

The dyadic maximal function $M_d(f)$ is defined by setting, for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$(4.3) \quad M_d(f)(x) := \sup_{k \in \mathbb{Z}} E_k(f)(x),$$

where, for any $k \in \mathbb{Z}$,

$$E_k(f)(x) := \sum_{Q \in \mathcal{Q}_k} \left[\frac{1}{|Q|} \int_Q |f(y)| dy \right] \chi_Q(x)$$

and $\mathcal{Q}_k := \{Q_\alpha^k : \alpha \in I_k\}$ denotes the set of dyadic cubes from Lemma 3.3. Moreover, by [15, Proposition A.4(ii)], we know that $f \leq M_d(f)$ almost everywhere.

By a slight modification on the proof of [15, Proposition A.5], we easily find that the conclusions of [15, Proposition A.5] also hold true for all $f \in L^{p,q}(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $q \in (0, \infty]$, the details being omitted. This provides the Calderón-Zygmund decomposition in the present setting, which is stated as follows.

Lemma 4.4. *Let $p \in (1, \infty)$, $q \in (0, \infty]$ and $f \in L^{p,q}(\mathbb{R}^n)$. Then, for all $\lambda \in (0, \infty)$, there exists a sequence $\{Q_j\}_j \subset \mathcal{Q}$ of mutually disjoint dyadic cubes such that*

- (i) $\bigcup_j Q_j = \{x \in \mathbb{R}^n : M_d(f)(x) > \lambda\}$;
- (ii) $|f(x)| \leq \lambda$ for almost every $x \notin \bigcup_j Q_j$;
- (iii) there exists a constant $C \in (1, \infty)$, independent of f and λ , such that

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq C\lambda;$$

- (iv) for any $Q \in \{Q_j\}_j$, there exists a unique $\widetilde{Q} \in \mathcal{Q}$ such that

$$Q \subset \widetilde{Q}, \quad \ell(Q) = \ell(\widetilde{Q}) - 1 \quad \text{and} \quad \frac{1}{|\widetilde{Q}|} \int_{\widetilde{Q}} |f(x)| dx < \lambda,$$

where \mathcal{Q} is as in Lemma 3.3 and $\ell(Q)$ is the level of Q .

Motivated by [73, Theorem 1], we obtain the following anisotropic Fefferman-Stein vector-valued inequality in $L^{p,q}(\mathbb{R}^n)$, which plays a key role in the proof of Theorem 2.12.

Lemma 4.5. *Let $r \in (1, \infty]$.*

- (i) *If $p \in (1, \infty)$ and $q \in (0, \infty]$, then there exists a positive constant C_9 such that, for all sequences $\{f_j\}_j$ of measurable functions,*

$$(4.4) \quad \left\| \left\{ \sum_j [M_{\text{HL}}(f_j)]^r \right\}^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} \leq C_9 \left\| \left[\sum_j |f_j|^r \right]^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} ;$$

- (ii) *If $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in (0, \min\{r, p\})$, then there exists a positive constant C_{10} such that, for all sequences $\{f_j\}_j$ of measurable functions,*

$$(4.5) \quad \left\| \left\{ \sum_j [M_{\text{HL}}(f_j)]^{\frac{r}{s}} \right\}^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} \leq C_{10} \left\| \left[\sum_j |f_j|^{\frac{r}{s}} \right]^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} .$$

Proof. Assume that the right-hand sides of (4.4) and (4.5) are finite. We first prove (i). To this end, let $f := [\sum_j |f_j|^r]^{\frac{1}{r}}$ and $\Omega_k := \{x \in \mathbb{R}^n : M_d(f)(x) > 2^k\}$ for any $k \in \mathbb{Z}$, where $M_d(f)$ is as in (4.3). By (i) and (iii) of Lemma 4.4, we obtain a sequence $\{Q_i^k\}_i$ of dyadic cubes satisfying that $\Omega_k = \bigcup_i Q_i^k$,

$$(4.6) \quad Q_i^k \cap Q_j^k = \emptyset \quad \text{for all } i, j \text{ with } i \neq j$$

and, for all i ,

$$(4.7) \quad \frac{1}{|Q_i^k|} \int_{Q_i^k} f(x) dx \lesssim 2^k.$$

For any j , let $f_j^{(1)} := f_j \chi_{\Omega_k}$ and $f_j^{(2)} := f_j \chi_{(\Omega_k)^c}$. Then it is easy to see that there exists a positive constant $C_{(p,q,r)}$, depending on p, q, r , but independent of $\{f_j\}_j$, such that

$$(4.8) \quad \begin{aligned} \left\| \left\{ \sum_j [M_{\text{HL}}(f_j)]^r \right\}^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} &\leq C_{(p,q,r)} \left[\left\| \left\{ \sum_j [M_{\text{HL}}(f_j^{(1)})]^r \right\}^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} \right. \\ &\quad \left. + \left\| \left\{ \sum_j [M_{\text{HL}}(f_j^{(2)})]^r \right\}^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)} \right] \\ &=: C_{(p,q,r)} (I_1 + I_2) . \end{aligned}$$

For I_1 , by (2.1), Lemma 4.1(ii), the fact $\Omega_k = \bigcup_i Q_i^k$, (4.7), (4.6) and [51, Remark 4.8], we have

$$\begin{aligned}
(4.9) \quad (I_1)^q &\sim \sum_{k \in \mathbb{Z}} 2^{kq} \left| \left\{ x \in \mathbb{R}^n : \left[\sum_j \left\{ M_{\text{HL}}(f_j^{(1)}) \right\}^r(x) \right]^{\frac{1}{r}} > 2^k \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-\frac{kq}{p}} \left\| \left[\sum_j |f_j^{(1)}|^r \right]^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}^{\frac{q}{p}} \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-\frac{kq}{p}} \left\{ \int_{\bigcup_i Q_i^k} f(x) dx \right\}^{\frac{q}{p}} \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-\frac{kq}{p}} \left[\sum_i 2^k |Q_i^k| \right]^{\frac{q}{p}} \lesssim \sum_{k \in \mathbb{Z}} 2^{kq} |\Omega_k|^{\frac{q}{p}} \lesssim \|M_d(f)\|_{L^{p,q}(\mathbb{R}^n)}^q \\
&\lesssim \|M_{\text{HL}}(f)\|_{L^{p,q}(\mathbb{R}^n)}^q \lesssim \|f\|_{L^{p,q}(\mathbb{R}^n)}^q.
\end{aligned}$$

For I_2 , take $m \in \mathbb{N}$ satisfying that $mr > p$. Then, by (2.1) and Lemma 4.1(i), we find that

$$\begin{aligned}
(4.10) \quad (I_2)^q &\sim \sum_{k \in \mathbb{Z}} 2^{kq} \left| \left\{ x \in \mathbb{R}^n : \left[\sum_j \left\{ M_{\text{HL}}(f_j^{(2)}) \right\}^r(x) \right]^{\frac{1}{r}} > 2^k \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-kmr \frac{q}{p}} \left[\int_{\mathbb{R}^n} \left\{ \sum_j \left[M_{\text{HL}}(f_j^{(2)}) \right]^r(x) \right\}^{\frac{1}{r} mr} dx \right]^{\frac{q}{p}} \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-kmr \frac{q}{p}} \left[\sum_{\ell \in \mathbb{Z}} 2^{\ell mr} \left| \left\{ x \in (\Omega_k)^c : f(x) > 2^\ell \right\} \right| \right]^{\frac{q}{p}}.
\end{aligned}$$

Next we estimate I_2 by considering two cases: $q/p \in (0, 1]$ and $q/p \in (1, \infty]$.

Case 1: $q/p \in (0, 1]$. For this case, by (4.10), the fact that $f(x) \leq M_d(f)(x) \leq 2^k$ for almost every $x \in (\Omega_k)^c$, $mr > p$ and (2.1), we know that

$$\begin{aligned}
(4.11) \quad (I_2)^q &\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-kmr \frac{q}{p}} \sum_{\ell \in (-\infty, k-1] \cap \mathbb{Z}} 2^{\ell mr \frac{q}{p}} \left| \left\{ x \in (\Omega_k)^c : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{\ell \in \mathbb{Z}} \sum_{k \in [\ell+1, \infty) \cap \mathbb{Z}} 2^{q(\ell-k)(\frac{mr}{p}-1)} 2^{\ell q} \left| \left\{ x \in (\Omega_k)^c : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{\ell \in \mathbb{Z}} 2^{\ell q} \left| \left\{ x \in \mathbb{R}^n : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \sim \|f\|_{L^{p,q}(\mathbb{R}^n)}^q.
\end{aligned}$$

Case 2: $q/p \in (1, \infty]$. For this case, let $\delta := \frac{mr-p}{2}$. Then, by (4.10), the Hölder inequality, the fact that $f(x) \leq M_d(f)(x) \leq 2^k$ for almost every $x \in (\Omega_k)^c$, $mr - \delta > p$

and (2.1) again, we have

$$\begin{aligned}
(I_2)^q &\lesssim \sum_{k \in \mathbb{Z}} 2^{kq} 2^{-kmr \frac{q}{p}} 2^{k\delta \frac{q}{p}} \sum_{\ell \in (-\infty, k-1] \cap \mathbb{Z}} 2^{\ell(mr-\delta) \frac{q}{p}} \left| \left\{ x \in (\Omega_k)^c : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{\ell \in \mathbb{Z}} \sum_{k \in [\ell+1, \infty) \cap \mathbb{Z}} 2^{q(\ell-k)(\frac{mr-\delta}{p}-1)} 2^{\ell q} \left| \left\{ x \in (\Omega_k)^c : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{\ell \in \mathbb{Z}} 2^{\ell q} \left| \left\{ x \in \mathbb{R}^n : f(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \sim \|f\|_{L^{p,q}(\mathbb{R}^n)}^q,
\end{aligned}$$

which, combined with (4.8), (4.9) and (4.11), implies (4.4). This prove (i).

Now we prove (ii). For any $p \in (0, \infty)$, $r \in (1, \infty)$ and $s \in (0, \min\{r, p\})$, by (2.3) and (i), we know that

$$\begin{aligned}
\left\| \left\{ \sum_j [M_{\text{HL}}(f_j)]^{\frac{r}{s}} \right\} \right\|_{L^{p,q}(\mathbb{R}^n)}^{\frac{1}{r}} &= \left\| \left\{ \sum_j [M_{\text{ML}}f_j]^{\frac{r}{s}} \right\} \right\|_{L^{p/s, q/s}(\mathbb{R}^n)}^{\frac{\frac{s}{r}}{\frac{1}{s}}} \\
&\lesssim \left\| \left[\sum_j |f_j|^{\frac{r}{s}} \right]^{\frac{s}{r}} \right\|_{L^{p/s, q/s}(\mathbb{R}^n)}^{\frac{1}{s}} \sim \left\| \left[\sum_j |f_j|^{\frac{r}{s}} \right]^{\frac{1}{r}} \right\|_{L^{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

This finishes the proof of (ii) and hence Lemma 4.5. \square

Definition 4.6. Let $r, \lambda \in (0, \infty)$. For any sequence $\{s_Q\}_{Q \in \widehat{\mathcal{Q}}} \subset \mathbb{C}$, its *majorant sequence* $s_{r,\lambda}^* := \{(s_{r,\lambda}^*)_Q\}_{Q \in \widehat{\mathcal{Q}}}$, is defined by setting, for all $Q \in \widehat{\mathcal{Q}}$,

$$(s_{r,\lambda}^*)_Q := \left\{ \sum_{P \in \widehat{\mathcal{Q}}, |P|=|Q|} \frac{|s_P|^r}{[1 + |Q|^{-1} \rho(x_Q - x_P)]^\lambda} \right\}^{\frac{1}{r}}.$$

The following lemma is just [12, Lemma 6.2].

Lemma 4.7. Let $a \in (0, \infty)$, $r \in [a, \infty)$, $\lambda \in (r/a, \infty)$ and $i, j \in \mathbb{Z}$. Then there exists a positive constant C , depending only on $\lambda - r/a$, such that, for any sequence $s := \{s_P\}_{P \in \widehat{\mathcal{Q}}}$, $Q \in \widehat{\mathcal{Q}}$ with $|Q| = |\det A|^{-j}$ and $x \in Q$, it holds true that

$$\begin{aligned}
&\left\{ \sum_{|P|=|\det A|^{-i}} |s_P|^r \left[1 + \frac{\rho(x_Q - x_P)}{\max\{|P|, |Q|\}} \right]^{-\lambda} \right\}^{\frac{1}{r}} \\
&\leq C |\det A|^{\frac{\max\{0, i-j\}}{a}} \left\{ \left[M_{\text{HL}} \left(\sum_{|P|=|\det A|^{-i}} |s_P|^a \chi_P \right) \right] (x) \right\}^{\frac{1}{a}}
\end{aligned}$$

and, in particular, if $i = j$, then

$$\sum_{|Q|=|\det A|^{-j}} (s_{r,\lambda}^*)_Q \chi_Q \leq C \left[M_{\text{HL}} \left(\sum_{|Q|=|\det A|^{-j}} |s_Q|^a \chi_Q \right) \right]^{\frac{1}{a}}.$$

Lemma 4.8. *Let $p \in (0, \infty)$ and $q \in (0, \infty]$. Then, for any $r \in (0, \infty)$ and $\lambda \in (\max\{1, r/2, r/p\}, \infty)$, there exists a positive constant C such that, for all $s := \{s_Q\}_{Q \in \widehat{\mathcal{Q}}}$,*

$$\left\| \left[\sum_{Q \in \widehat{\mathcal{Q}}} \left\{ (s_{r,\lambda}^*)_Q \right\}^2 \chi_Q \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \leq C \left\| \left[\sum_{Q \in \widehat{\mathcal{Q}}} |s_Q|^2 \chi_Q \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)}$$

Proof. Let $r \in (0, \infty)$ and $\lambda \in (\max\{1, r/2, r/p\}, \infty)$. Choose a such that $r/\lambda < a < \min\{r, 2, p\}$. Then $0 < a < r < \infty$, $\lambda > r/a$, $2/a > 1$ and $p/a > 1$. Therefore, by Lemma 4.7, (2.3) and Lemma 4.5(i), we find that

$$\begin{aligned} & \left\| \left[\sum_{Q \in \widehat{\mathcal{Q}}} \left\{ (s_{r,\lambda}^*)_Q \right\}^2 \chi_Q \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\ &= \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{|Q|=|\det A|^{-j}} (s_{r,\lambda}^*)_Q \chi_Q \right]^2 \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\ &\lesssim \left\| \left(\left\{ \sum_{j \in \mathbb{Z}} \left[M_{\text{HL}} \left(\sum_{|Q|=|\det A|^{-j}} |s_Q|^a \chi_Q \right) \right]^{\frac{2}{a}} \right\}^{\frac{a}{2}} \right)^{\frac{1}{a}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\ &\sim \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[M_{\text{HL}} \left(\sum_{|Q|=|\det A|^{-j}} |s_Q|^a \chi_Q \right) \right]^{\frac{2}{a}} \right\}^{\frac{a}{2}} \right\|_{L^{p/a, q/a}(\mathbb{R}^n)}^{\frac{1}{a}} \\ &\lesssim \left\| \left[\sum_{j \in \mathbb{Z}} \left(\sum_{|Q|=|\det A|^{-j}} |s_Q|^a \chi_Q \right)^{\frac{2}{a}} \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \sim \left\| \left[\sum_{Q \in \widehat{\mathcal{Q}}} |s_Q|^2 \chi_Q \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of Lemma 4.8. \square

The following Lemma 4.9 comes from [46, Lemma 3.8] (see also [11, p. 423]).

Lemma 4.9. *For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\text{supp } \widehat{\Phi}$ being compact and bounded away from the origin, the sequences $\sup(f) := \{\sup_Q(f)\}_{Q \in \widehat{\mathcal{Q}}}$ and $\inf(f) := \{\inf_Q(f)\}_{Q \in \widehat{\mathcal{Q}}}$ are defined by setting, respectively, for any $Q \in \widehat{\mathcal{Q}}$ with $|Q| = |\det A|^{-j}$ for some $j \in \mathbb{Z}$,*

$$\sup_Q(f) := \sup_{y \in Q} |f * \widetilde{\Phi}_{-j}(y)|$$

and

$$\inf_Q(f) := \sup \left\{ \inf_{y \in P} |f * \widetilde{\Phi}_{-j}(y)| : |P \cap Q| \neq 0, |Q|/|P| = |\det A|^\gamma \right\},$$

where $\gamma \in \mathbb{N}$. Then, for all $\lambda, r \in (0, \infty)$ and sufficient large $\gamma \in \mathbb{N}$, there exists a positive constant C such that, for any $Q \in \widehat{\mathcal{Q}}$,

$$(\sup_Q(f))_{r, \lambda}^* \leq C(\inf_Q(f))_{r, \lambda}^*.$$

Now we prove Theorem 2.12.

Proof of Theorem 2.12. We first prove the necessity of Theorem 2.12. Let $p \in (0, 1]$, $q \in (0, \infty]$ and $f \in H_A^{p, q}(\mathbb{R}^n)$. By Proposition 3.1, we know $f \in \mathcal{S}'_0(\mathbb{R}^n)$. Furthermore, by [46, Lemma 2.20] and repeating the proof of the necessity of Theorem 2.11 with a slight modification, we easily conclude that $g(f) \in L^{p, q}(\mathbb{R}^n)$ and $\|g(f)\|_{L^{p, q}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p, q}(\mathbb{R}^n)}$. This finishes the proof of the necessity of Theorem 2.12.

Thus, to complete the proof of Theorem 2.12, it remains to show the sufficiency of Theorem 2.12. To this end, by Theorem 2.11, we only need to prove that

$$(4.12) \quad \|S(f)\|_{L^{p, q}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p, q}(\mathbb{R}^n)}$$

for all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ with $g(f) \in L^{p, q}(\mathbb{R}^n)$. We prove this by two steps.

Step 1. In this step, we prove that, for any $r \in (0, 1]$, $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $y \in x + B_{-j}$,

$$(4.13) \quad |f * \varphi_{-j}(y)| \lesssim \sum_{i \in \mathbb{Z}} b^{-|j-i|[1+(s+1)\zeta_- - \frac{1}{r}]}$$

$$\times \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \widetilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r},$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has the vanishing moments up to ℓ which will be fixed later.

To this end, let $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ be as in Lemma 4.3. For any $M \in \mathbb{N}$, $j \in \mathbb{Z}$, $f \in \mathcal{S}'_0(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $y \in x + B_{-j}$, by Lemmas 4.3 and 4.2, we have

$$|f * \varphi_{-j}(y)|$$

$$\lesssim \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_i} b^{-i} \left| f * \widetilde{\Phi}_{-i}(x_Q) \Psi_{-i} * \varphi_{-j}(y - x_Q) \right|$$

$$\lesssim \left[\sum_{i \leq j} \sum_{Q \in \mathcal{Q}_i} \frac{b^{-(j-i)(\ell+1)\zeta_-}}{[1 + \rho(A^i(y - x_Q))]^M} + \sum_{i > j} \sum_{Q \in \mathcal{Q}_i} \frac{b^{-(i-j)[1+(\ell+1)\zeta_-]}}{[1 + \rho(A^j(y - x_Q))]^M} \right] \left| (f * \widetilde{\Phi}_{-i})(x_Q) \right|$$

$$=: C_{11}(\text{I} + \text{II}),$$

where C_{11} is a positive constant depending on ℓ and M , but independent of j , f and y .

We first estimate I. Assume that $i \leq j$. For any $k \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$ and $y \in x + B_{-j}$, when $k = 0$, let $U_0 := \{Q \in \mathcal{Q}_i : \rho(A^i(y - x_Q)) \leq 1\}$ and, when $k \in \mathbb{N}$, let

$$U_k := \{Q \in \mathcal{Q}_i : b^{k-1} < \rho(A^i(y - x_Q)) \leq b^k\}.$$

Then we have

$$(4.14) \quad \begin{aligned} & \sum_{Q \in U_k} \frac{b^{-(j-i)(\ell+1)\zeta_-}}{[1 + \rho(A^i(y - x_Q))]^M} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right| \\ & \lesssim b^{-(j-i)(\ell+1)\zeta_- - kM} \sum_{Q \in U_k} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|. \end{aligned}$$

Notice that, for any $z \in Q \in U_k$, by Definition 2.2(iii) and (4.1), we know that

$$\rho(z - y) \leq H[\rho(z - x_Q) + \rho(x_Q - y)] \leq H(b^{j_0-i} + b^{k-i}) < 2Hb^{j_0+k-i},$$

where $j_0 \in \mathbb{Z}_+$ is as in (4.1), which implies that

$$(4.15) \quad \bigcup_{Q \in U_k} Q \subset B_\rho(y, 2Hb^{j_0+k-i}) := \{z \in \mathbb{R}^n : \rho(y - z) < 2Hb^{j_0+k-i}\}.$$

Moreover, noticing that $i \leq j$ and $k, j_0 \in \mathbb{Z}_+$, for all $x \in y + B_{-j}$, we have $x \in B_\rho(y, 2Hb^{j_0+k-i})$. Therefore, for all $r \in (0, 1]$ and $x \in y + B_{-j}$, by (4.15), we conclude that

$$(4.16) \quad \begin{aligned} & \sum_{Q \in U_k} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right| \\ & \lesssim b^{\frac{k}{r}} \left\{ \frac{1}{|B_\rho(y, 2Hb^{j_0+k-i})|} \int_{B_\rho(y, 2Hb^{j_0+k-i})} \sum_{Q \in U_k} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q(z) dz \right\}^{1/r} \\ & \lesssim b^{\frac{k}{r}} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r}. \end{aligned}$$

We choose $M > 1/r$. Then, by (4.14) and (4.16), we find that

$$\begin{aligned} \text{I} & \lesssim \sum_{i \leq j} \sum_{k=0}^{\infty} b^{-(j-i)(\ell+1)\zeta_- - k(M - \frac{1}{r})} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r} \\ & \sim \sum_{i \leq j} b^{-(j-i)(\ell+1)\zeta_-} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r}. \end{aligned}$$

On the other hand, similar to the estimate of I, by choosing $M, \ell \in \mathbb{N}$ such that $M > 1/r$ and $1 + (\ell + 1)\zeta_- - 1/r > 0$, we also obtain

$$\begin{aligned} \text{II} &\lesssim \sum_{i>j} \sum_{k=0}^{\infty} b^{-(i-j)[1+(\ell+1)\zeta_- - \frac{1}{r}] - k(M - \frac{1}{r})} \\ &\quad \times \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r} \\ &\sim \sum_{i>j} b^{-(i-j)[1+(\ell+1)\zeta_- - \frac{1}{r}]} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r}. \end{aligned}$$

Combining the above estimates of I and II, we further conclude that (4.13) holds true.

Step 2. In this step, we show (4.12) via (4.13). Indeed, by (4.13), we know that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} (4.17) \quad [S(f)(x)]^2 &= \sum_{j \in \mathbb{Z}} b^j \int_{x+B_{-j}} |f * \varphi_{-j}(y)|^2 dy \\ &\lesssim \sum_{j \in \mathbb{Z}} \left\{ \sum_{i \in \mathbb{Z}} b^{-|j-i|[1+(\ell+1)\zeta_- - \frac{1}{r}]} \right. \\ &\quad \times \left. \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{1/r} \right\}^2. \end{aligned}$$

Noticing that $1 + (\ell + 1)\zeta_- - 1/r > 0$, by the Hölder inequality, we have

$$\begin{aligned} &\sum_{i \in \mathbb{Z}} b^{-|j-i|[1+(\ell+1)\zeta_- - \frac{1}{r}]} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{\frac{1}{r}} \\ &\lesssim \left\{ \sum_{i \in \mathbb{Z}} b^{-|j-i|[1+(\ell+1)\zeta_- - \frac{1}{r}]} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{\frac{2}{r}} \right\}^{\frac{1}{2}}, \end{aligned}$$

which, combined with (4.17), further implies that

$$(4.18) \quad S(f)(x) \lesssim \left\{ \sum_{i \in \mathbb{Z}} \left\{ \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} \left| (f * \tilde{\Phi}_{-i})(x_Q) \right|^r \chi_Q \right) \right] (x) \right\}^{\frac{2}{r}} \right\}^{\frac{1}{2}}.$$

Choose $M \in \mathbb{N}$ large enough such that $r \in (1/M, p)$. Then, by (4.18) and Lemma 4.5(i),

we find that

$$\begin{aligned}
 (4.19) \quad \|S(f)\|_{L^{p,q}(\mathbb{R}^n)} &\lesssim \left\| \left\{ \sum_{i \in \mathbb{Z}} \left[M_{\text{HL}} \left(\sum_{Q \in \mathcal{Q}_i} |(f * \tilde{\Phi}_{-i})(x_Q)|^r \chi_Q \right) \right]^{\frac{2}{r}} \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\
 &\lesssim \left\| \left\{ \sum_{i \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_i} [|(f * \tilde{\Phi}_{-i})(x_Q)| \chi_Q]^2 \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)}.
 \end{aligned}$$

Recall that $s_Q \leq (s_{r,\lambda}^*)_Q$ for any sequence $\{s_Q\}_{Q \in \widehat{\mathcal{Q}}} \subset \mathbb{C}$, and $r, \lambda \in (0, \infty)$. From this, (4.19), Lemmas 4.9 and 4.8 with $r \in (0, \infty)$, $\lambda \in (\max\{1, r/2, r/p\}, \infty)$, we deduce that, for some $\gamma \in \mathbb{N}$ large enough as in Lemma 4.9,

$$\begin{aligned}
 (4.20) \quad \|S(f)\|_{L^{p,q}(\mathbb{R}^n)} &\lesssim \left\| \left\{ \sum_{Q \in \widehat{\mathcal{Q}}} [(\sup_Q(f))_{r,\lambda}^*]^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\
 &\lesssim \left\| \left\{ \sum_{Q \in \widehat{\mathcal{Q}}} [(\inf_Q(f))_{r,\lambda}^*]^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\
 &\lesssim \left\| \left\{ \sum_{Q \in \widehat{\mathcal{Q}}} [\inf_Q(f)]^2 \chi_Q \right\}^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)}.
 \end{aligned}$$

Moreover, for any $P \in \widehat{\mathcal{Q}}$ with $|P| = b^{-i}$ and $s_P := \inf_{y \in P} |f * \tilde{\Phi}_{i-\gamma}(y)|$, it follows from [46, p. 306] that $\inf_Q(f) = \sup\{s_P : P \in \widehat{\mathcal{Q}}, P \cap Q \neq \emptyset, |Q|/|P| = b^\gamma\}$ and, for all $x \in \mathbb{R}^n$,

$$\sum_{|Q|=b^{-j}} \inf_Q(f) \chi_Q(x) \lesssim b^{\frac{\gamma\lambda}{r}} \sum_{|P|=b^{-j-\gamma}} (s_{r,\lambda}^*)_P \chi_P(x)$$

(see also the proof of [11, Lemma 8.4] for more details). By this, (4.20) and Lemma 4.8, we have

$$\begin{aligned}
 \|S(f)\|_{L^{p,q}(\mathbb{R}^n)} &\lesssim \left\| \left[\sum_{j \in \mathbb{Z}} \sum_{|P|=b^{-j-\gamma}} |(s_{r,\lambda}^*)_P|^2 \chi_P \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \\
 &\lesssim \left\| \left[\sum_{i \in \mathbb{Z}} \sum_{|P|=b^{-i}} |s_P|^2 \chi_P \right]^{\frac{1}{2}} \right\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p,q}(\mathbb{R}^n)},
 \end{aligned}$$

which is (4.12). This finishes the proof of the sufficiency of Theorem 2.12 and hence Theorem 2.12. \square

To prove Theorem 2.13, we first recall some notation from [46]. For each open subset $E \subset \mathbb{R}^n$ and $k_0 \in \mathbb{N}$, let

$$(4.21) \quad U_{k_0} := \left\{ x \in \mathbb{R}^n : M_{\text{HL}}(\chi_E)(x) > b^{-2\tau-k_0} \right\},$$

where M_{HL} denotes the Hardy-Littlewood maximal function defined by (2.11) and $\tau \in \mathbb{Z}_+$ is as in (2.6). For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $k_0 \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_{k_0}(f)(x) := \left\{ \sum_{k \in \mathbb{Z}} b^{-(k+k_0)} \int_{x+B_{k+k_0}} |f * \varphi_k(y)|^2 dy \right\}^{\frac{1}{2}},$$

which is a variant of the anisotropic Lusin-area function S defined by (2.12). Obviously, $S = S_0$.

The proof of the following Lemma 4.10 is similar to that of [46, Lemma 3.12], the details being omitted.

Lemma 4.10. *There exists a positive constant C such that, for all $k_0 \in \mathbb{N}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\int_{(U_{k_0})^c} [S_{k_0}(f)(x)]^2 dx \leq C \int_{E^c} [S(f)(x)]^2 dx,$$

where $E \subset \mathbb{R}^n$ is an open set and U_{k_0} is as in (4.21).

The following technical lemma plays a key role in proving Theorem 2.13, whose proof is motivated by Folland and Stein [36, Theorem 7.1], Aguilera and Segovia [2, Theorem 1], Liang et al. [47, Lemma 4.6] and Li et al. [46, Lemma 3.13].

Lemma 4.11. *Let $p \in (0, 1]$ and $q \in (0, \infty]$. Then there exists a positive constant C such that, for all $k_0 \in \mathbb{N}$ and $f \in L^{p,q}(\mathbb{R}^n)$,*

$$\|S_{k_0}(f)\|_{L^{p,q}(\mathbb{R}^n)} \leq C b^{(\frac{1}{p}-\frac{1}{2})k_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

Proof. For any $k_0 \in \mathbb{N}$, $\lambda \in (0, \infty)$ and $f \in L^{p,q}(\mathbb{R}^n)$, let

$$E_{\lambda,k_0} := \left\{ x \in \mathbb{R}^n : S(f)(x) > \lambda b^{k_0/2} \right\}$$

and

$$U_{\lambda,k_0} := \left\{ x \in \mathbb{R}^n : M_{\text{HL}}(\chi_{E_{\lambda,k_0}})(x) > b^{-2\tau-k_0} \right\},$$

where M_{HL} is as in (2.11). Then, by the boundedness from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ of M_{HL} (see Proposition 2.4 and Remark 2.5), we have

$$|U_{\lambda,k_0}| \lesssim b^{k_0} \|\chi_{E_{\lambda,k_0}}\|_{L^1(\mathbb{R}^n)} \sim b^{k_0} |E_{\lambda,k_0}|.$$

From this, together with Lemma 4.10 with $E = E_{\lambda,k_0}$ and $U_{k_0} = U_{\lambda,k_0}$, we deduce that

$$(4.22) \quad |\{x \in \mathbb{R}^n : S_{k_0}(f) > \lambda\}|$$

$$\begin{aligned}
&\leq |U_{\lambda,k_0}| + \left| (U_{\lambda,k_0})^c \cap \{x \in \mathbb{R}^n : S_{k_0}(f)(x) > \lambda\} \right| \\
&\lesssim b^{k_0} |E_{\lambda,k_0}| + \lambda^{-2} \int_{(U_{\lambda,k_0})^c} [S_{k_0}(f)(x)]^2 dx \\
&\lesssim b^{k_0} |E_{\lambda,k_0}| + \lambda^{-2} \int_{(E_{\lambda,k_0})^c} [S(f)(x)]^2 dx \\
&\sim b^{k_0} |E_{\lambda,k_0}| + \lambda^{-2} \int_0^{\lambda b^{k_0/2}} \mu |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}| d\mu.
\end{aligned}$$

Moreover, for all $\ell \in \mathbb{Z}$ and $k_0 \in \mathbb{N}$, by (4.22), we have

$$\begin{aligned}
(4.23) \quad &\left| \{x \in \mathbb{R}^n : S_{k_0}(f)(x) > 2^\ell\} \right| \\
&\lesssim b^{k_0} |E_{2^\ell,k_0}| + 2^{-2\ell} \int_0^{2^\ell b^{k_0/2}} \mu |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}| d\mu \\
&\lesssim b^{k_0} |E_{2^\ell,k_0}| + 2^{-2\ell} \sum_{m=-\infty}^{m_\ell} \int_{2^{m-1}}^{2^m} \mu |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}| d\mu \\
&\lesssim b^{k_0} |E_{2^\ell,k_0}| + 2^{-2\ell} \sum_{m=-\infty}^{m_\ell} 2^{2(m-1)} |\{x \in \mathbb{R}^n : S(f)(x) > 2^{m-1}\}|,
\end{aligned}$$

where $m_\ell := \ell + \lfloor \frac{k_0}{2} \log_2 b \rfloor + 1$. Next we show the desired conclusion by considering three cases: $q/p \in (1, \infty)$, $q/p \in (0, 1]$ and $q = \infty$.

Case 1: $q/p \in (1, \infty)$. For this case, by (2.1), (4.22), the definition of E_{λ,k_0} and the Hölder inequality, we conclude that

$$\begin{aligned}
(4.24) \quad &\|S_{k_0}(f)\|_{L^{p,q}(\mathbb{R}^n)}^q \sim \int_0^\infty \lambda^{q-1} |\{x \in \mathbb{R}^n : S_{k_0}(f)(x) > \lambda\}|^{\frac{q}{p}} d\lambda \\
&\lesssim \int_0^\infty b^{\frac{q}{p}k_0} \lambda^{q-1} \left| \{x \in \mathbb{R}^n : S(f)(x) > \lambda b^{k_0/2}\} \right|^{\frac{q}{p}} d\lambda \\
&\quad + \int_0^\infty \lambda^{q-\frac{2q}{p}-1} \left[\int_0^{\lambda b^{k_0/2}} \mu |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}| d\mu \right]^{\frac{q}{p}} d\lambda \\
&\lesssim \int_0^\infty b^{(\frac{1}{p}-\frac{1}{2})qk_0} \lambda^{q-1} |\{x \in \mathbb{R}^n : S(f)(x) > \lambda\}|^{\frac{q}{p}} d\lambda \\
&\quad + \int_0^\infty \lambda^{q-\frac{2q}{p}-1} \left[\int_0^{\lambda b^{k_0/2}} \mu^{(1-p)\frac{q}{q-p}} d\mu \right]^{\frac{q}{p}-1} \\
&\quad \times \int_0^{\lambda b^{k_0/2}} \mu^q |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}|^{\frac{q}{p}} d\mu d\lambda \\
&\sim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q + b^{(\frac{1}{p}-\frac{1}{2})qk_0-\frac{k_0}{2}} \\
&\quad \times \int_0^\infty \lambda^{-2} \int_0^{\lambda b^{k_0/2}} \mu^q |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}|^{\frac{q}{p}} d\mu d\lambda
\end{aligned}$$

$$\begin{aligned}
&\lesssim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q + b^{(\frac{1}{p}-\frac{1}{2})qk_0} \\
&\quad \times \int_0^\infty \mu^{q-1} |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}|^{\frac{q}{p}} d\mu \\
&\sim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q.
\end{aligned}$$

Case 2: $q/p \in (0, 1]$. For this case, by (2.1), (4.23) and the definition of E_{λ, k_0} , we find that

$$\begin{aligned}
(4.25) \quad &\|S_{k_0}(f)\|_{L^{p,q}(\mathbb{R}^n)}^q \sim \sum_{\ell \in \mathbb{Z}} 2^{\ell q} \left| \left\{ x \in \mathbb{R}^n : S_{k_0}(f)(x) > 2^\ell \right\} \right|^{\frac{q}{p}} \\
&\lesssim \sum_{\ell \in \mathbb{Z}} 2^{\ell q} b^{\frac{q}{p}k_0} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^\ell b^{k_0/2} \right\} \right|^{\frac{q}{p}} \\
&\quad + \sum_{\ell \in \mathbb{Z}} 2^{\ell q} 2^{-\frac{2q}{p}\ell} \sum_{m=-\infty}^{m_\ell} 2^{2(m-1)\frac{q}{p}} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^{m-1} \right\} \right|^{\frac{q}{p}} \\
&\lesssim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q + \sum_{\ell \in \mathbb{Z}} 2^{-\ell q} b^{(\frac{1}{p}-1)qk_0} \\
&\quad \times \sum_{m=-\infty}^{m_\ell} 2^{2q(m-1)} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^{m-1} \right\} \right|^{\frac{q}{p}} \\
&\sim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q + \sum_{m \in \mathbb{Z}} 2^{-\ell q} b^{(\frac{1}{p}-1)qk_0} \\
&\quad \times \sum_{\ell=\ell_m}^{\infty} 2^{2q(m-1)} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^{m-1} \right\} \right|^{\frac{q}{p}} \\
&\lesssim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q + b^{(\frac{1}{p}-\frac{1}{2})qk_0} \\
&\quad \times \sum_{m \in \mathbb{Z}} 2^{q(m-1)} \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > 2^{m-1} \right\} \right|^{\frac{q}{p}} \\
&\sim b^{(\frac{1}{p}-\frac{1}{2})qk_0} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^q,
\end{aligned}$$

where $\ell_m := m - \lfloor \frac{k_0}{2} \log_2 b \rfloor - 1$.

Case 3: For the last case when $q = \infty$, by (2.2), (4.22) and the definition of E_{λ, k_0} again, we know that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}
(4.26) \quad &\lambda |\{x \in \mathbb{R}^n : S_{k_0}(f)(x) > \lambda\}|^{\frac{1}{p}} \\
&\lesssim b^{\frac{k_0}{p}} \lambda \left| \left\{ x \in \mathbb{R}^n : S(f)(x) > b^{\frac{k_0}{2}} \lambda \right\} \right|^{\frac{1}{p}} \\
&\quad + \lambda^{1-\frac{2}{p}} \left\{ \int_0^{\lambda b^{k_0/2}} \mu |\{x \in \mathbb{R}^n : S(f)(x) > \mu\}| d\mu \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|S(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \left[b^{(\frac{1}{p}-\frac{1}{2})k_0} + \lambda^{1-\frac{2}{p}} \left(\int_0^{\lambda b^{k_0/2}} \mu^{1-p} d\mu \right)^{\frac{1}{p}} \right] \\
&\sim b^{(\frac{1}{p}-\frac{1}{2})k_0} \|S(f)\|_{L^{p,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Combining (4.24), (4.25) and (4.26), we complete the proof of Lemma 4.11. \square

Now we prove Theorem 2.13.

Proof of Theorem 2.13. Let $p \in (0, 1]$, $q \in (0, \infty]$ and $\lambda \in (2/p, \infty)$. Notice that, for all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ with $g_\lambda^*(f) \in L^{p,q}(\mathbb{R}^n)$, $S(f) \lesssim g_\lambda^*(f)$. Then, by Theorem 2.11, we know that

$$\|f\|_{H_A^{p,q}(\mathbb{R}^n)} \lesssim \|g_\lambda^*\|_{L^{p,q}(\mathbb{R}^n)}.$$

Conversely, for all $f \in H_A^{p,q}(\mathbb{R}^n)$, by Proposition 3.1, we know $f \in \mathcal{S}'_0(\mathbb{R}^n)$. Thus, to complete the proof of Theorem 2.13, by Theorem 2.11 again, we only need to show that, for all $f \in H_A^{p,q}(\mathbb{R}^n)$,

$$(4.27) \quad \|g_\lambda^*(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}.$$

Indeed, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
[g_\lambda^*(f)(x)]^2 &= \sum_{k \in \mathbb{Z}} b^{-k} \int_{x+B_k} \left[\frac{b^k}{b^k + \rho(x-y)} \right]^\lambda |f * \varphi_k(y)|^2 dy \\
&\quad + \sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}} b^{-k} \int_{x+(B_{k+m} \setminus B_{k+m-1})} \dots \\
&\lesssim [S(f)(x)]^2 + \sum_{m=1}^{\infty} b^{-m(\lambda-1)} [S_m(f)(x)]^2.
\end{aligned}$$

From this, the Aoki-Rolewicz theorem (see [7, 61]), Lemma 4.11 and $\lambda \in (2/p, \infty)$, we further deduce that there exists $v \in (0, 1]$ such that

$$\begin{aligned}
\|g_\lambda^*(f)\|_{L^{p,q}(\mathbb{R}^n)}^v &\lesssim \sum_{m=0}^{\infty} b^{-mv(\lambda-1)/2} \|S_m(f)\|_{L^{p,q}(\mathbb{R}^n)}^v \\
&\lesssim \sum_{m=0}^{\infty} b^{-mv(\lambda-1)/2} b^{(\frac{1}{p}-\frac{1}{2})mv} \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^v \lesssim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}^v,
\end{aligned}$$

which implies that $\|g_\lambda^*(f)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p,q}(\mathbb{R}^n)}$. This finishes the proof of (4.27) and hence Theorem 2.13. \square

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